Mathematical Reviews

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ANALYSIS

★Madelung, Erwin. Die Mathematischen Hilfsmittel des Physikers. Dover Publications, N. Y., 1943. viii+384 pp. $3.50.

Photoprint of the third (latest) German edition of 1936 [J. Springer, Berlin]. A German-English glossary of scientific and technical terms has been added to the present edition.

Theory of Sets, Theory of Functions of Real Variables

Riabouchinsky, Dimitri. Sur les définitions analytiques du continu. C. R. Acad. Sci. Paris **212**, 1109–1112 (1941). [MF 9185]

The reasoning of this note represents a reversion from modern rigor to the naive reasoning of a much earlier day.
H. Blumberg (Columbus, Ohio).

Denjoy, Arnaud. Sur les nombres transfinis. C. R. Acad. Sci. Paris **213**, 430–433 (1941). [MF 9167]

A permutation P of the set of all positive integers is defined as a system $[n, >]$, where $n>m$ is a transitive relation defined for every pair of distinct integers. Using an earlier result of his [C. R. Acad. Sci. Paris **212**, 885–888 (1941), in particular, p. 886; these Rev. **3**, 73], the author asserts a one to one correspondence between the Naperian developments of real numbers and the class of permutations P in terms of which he states a necessary and sufficient condition that a permutation P be well-ordered. The note concludes with remarks insisting on the distinction between the notions of well-ordered order type and ordinal number.
L. W. Cohen (Madison, Wis.).

Otchan, G. Sur la comparaison des puissances des opérations δs. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 186–190 (1941). [MF 9628]

Of two Hausdorff operations δs [see Hausdorff, Mengenlehre, Gruyter, Berlin, 1935, p. 89] one is said to be stronger than the other if, starting with an arbitrary family of sets, the family resulting from the application to it of the stronger operation contains the family resulting from the application of the other. L. Kantorovitch and Livenson [Fund. Math. **18**, 214–279 (1932)] have given a necessary and sufficient condition for this. The present paper is concerned with the same problem when the families of sets considered are restricted to certain particular ones. The precise results cannot be stated briefly.
J. V. Wehausen.

Otchan, G. Opération A généralisée. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 393–396 (1941). [MF 9551]

A "cortège" of type α is a set of transfinite numbers less than ω_α which forms a transfinite sequence whose type is a transfinite number with immediate predecessor and less than ω_α. With each cortège is associated some set $F_{n_1 n_2 \ldots n_\nu}$ of a class $\{F\}$ of sets contained in a set E. An A-set of type α is one of the form $\sum \prod F_{n_1 n_2 \ldots n_\nu}$, where the sum is over all sequences of numbers less than ω_α and the product over all cortèges of type α which are segments of some such sequence. With each transfinite number $i<\omega_\beta$ let there be associated a set F_i from a class $\{F\}$. Let ζ be a sequence of numbers less than ω_β and N a set of such sequences. Then $\sum_{\zeta\in N} \prod_{i\in\zeta} F_i$ is called a δs-operation of type β. These operations can now be used in the customary fashion to obtain extensions of the class $\{F\}$. A theorem, analogous to that of L. Kantorovitch and E. Livenson [Fund. Math. **18**, 214–279 (1932); in particular, p. 235] for $\beta=0$, concerning the relative strength of two such δs-operations of type β is proved. It is shown that an operation A of type α is a normal δs-operation of type β for some β ("normal" means that its iteration does not result in further class extension). A definition of B-measurable sets of type α over $\{F\}$ is also given and it is shown that such sets can be obtained by application of the operation A of type α to $\{F\}$.
J. V. Wehausen (Columbia, Mo.).

Rutt, N. E. On derived sets. Nat. Math. Mag. **18**, 53–63 (1943). [MF 9556]

This article is based on the following abstract conception of derived set, which discards from this notion all but the functional relationship. Let S be a given set, U the set of subsets of S and $e'=f(e)$ a given correspondence which mates with each element e of U a unique element e' of U; then e' is defined to be the derived set of e. To secure results, the following definition in introduced: e is closed of grade n ($n=1, 2, \cdots$) if the nth derived set (that is, the derived set of the $(n-1)$th derived set) of every subset of e is a subset of e. Illustrative results: if e_1 and e_2 are closed of grade n, then e_1e_2 is likewise; if $e_1, e_2, \cdots$ is an infinite descending sequence of closed sets of grade n, then the product of the e_ν is closed of grade n. There is no mention of literature applications.
H. Blumberg.

Ulam, S. M. What is measure? Amer. Math. Monthly **50**, 597–602 (1943). [MF 9849]

Expository article.

Hewitt, Edwin. Two notes on measure theory. Bull. Amer. Math. Soc. **49**, 719–721 (1943). [MF 9310]

The author proves first a theorem about the existence of a linear functional over the space C of all continuous real-valued functions on a compact metric space, non negative for non negative functions. Secondly, he proves a theorem asserting that on any infinite set there exists a measure function defined on all subsets of the set, completely additive and equal to 0 for all sets consisting of a single point except a countable set of such points. This theorem is really obvious as might be seen from considering a measure distributed in form of non negative weights with sum equal to 1 over an arbitrary countable set of points. It follows from a known theorem that a measure defined for all subsets must be of this form.
S. M. Ulam (Madison, Wis.).

Nöbeling, Georg. Über die Länge der Euklidischen Kontinuen. I. Jber. Deutsch. Math. Verein. **52**, 132–160 (1942). [MF 8541]

This paper contains a collection of thirteen definitions of linear measure for sets of Euclidean space of n-dimensions, together with proofs that for a continuum all of the linear measures listed are the same. [Cf. the following two reviews.] *J. F. Randolph* (Ithaca, N. Y.).

Nöbeling, Georg. Über die Länge der Euklidischen Kontinuen. II. Jber. Deutsch. Math. Verein. **52**, 189–197 (1942). [MF 9051]

[Cf. the preceding and the following reviews.] It is shown that for continua K contained in n-space there is a unique nonnegative function $\lambda(K)$, called length, satisfying the following axioms: (1) $\lambda(K) \leqq \lambda(K_1)+\cdots+\lambda(K_m)$ if $K \subset K_1+\cdots+K_m$; (2) $\lambda(K) \geqq \lambda(K_1)+\cdots+\lambda(K_m)$ if $K_1, \cdots, K_m$ are disjoint continua contained in K; (3) congruent continua have equal length; (4) $\lambda(K) \leqq \liminf \lambda(K_i)$ if $K \subset \liminf K_i$; (5) a unit segment has unit length. Another set of axioms, identical with the set due to Kolmogoroff [Math. Ann. **107**, 351–366 (1932)] except that the measure is assumed to be defined only for continua, is also shown to characterize $\lambda(K)$. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Nöbeling, Georg. Über die Länge der Euklidischen Kontinuen. III. Monatsh. Math. Phys. **50**, 282–287 (1942). [MF 8546]

[Cf. the preceding two reviews.] S. Banach obtained a necessary and sufficient condition that a plane curve be rectifiable in terms of the number of intersections of the curve by lines perpendicular to an axis [Fund. Math. **7**, 223–236 (1925)]. In this paper Banach's proof is generalized to obtain a necessary and sufficient condition that a continuum of Euclidean n-dimensional space have finite linear measure (under any of the definitions of part I).
J. F. Randolph (Ithaca, N. Y.).

Pauc, Christian. Über ebene Punktmengen, welche überall einen Sektor von gegebener Grösse freilassen. J. Reine Angew. Math. **185**, 127–128 (1943). [MF 9595]

A Lipschitz arc can be represented in suitable Cartesian coordinates x,y in the form $y=f(x)$, $(a \leqq x \leqq b)$, where $f(x)$ satisfies a Lipschitz condition

$$|f(x_1)-f(x_2)| \leqq \text{const. } |x_1-x_2|$$

for all the pairs of points x_1, x_2 of the interval $a \leqq x \leqq b$. Let $\theta>0$ be a given angle, $\rho>0$ a given radius. Let M be a bounded set in the Euclidean plane. The circle of radius ρ around each point of M shall contain a sector with the angle θ that contains no other points of M. Then M is a subset of a finite sum of Lipschitz arcs. *P. Scherk.*

Germeier, G. Sur les nombres dérivés symétriques. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 121–145 (1943). (Russian. French summary) [MF 8798]

Let $f(x)$ be a measurable function defined on a set E. The upper and lower symmetric derivatives of f on E are defined, respectively, as

$$\bar{D}f(x)=\limsup_{h\to 0}\frac{f(x+h)-f(x-h)}{h},$$

$$\underline{D}f(x)=\liminf_{h\to 0}\frac{f(x+h)-f(x-h)}{h},$$

$x+h\epsilon E$, $x-h\epsilon E$. It is then proved that at almost every point either f has a finite first derivative or $\bar{D}f=+\infty$ and $\underline{D}f=-\infty$. Theorems of similar character for Schwarz's derivatives are also proved. *M. Kac* (Ithaca, N. Y.).

Roger, Frédéric. Sur un problème de M. Denjoy. C. R. Acad. Sci. Paris **214**, 942–944 (1942). [MF 9468]

If the "differential coefficients" $f_1(x_0), \cdots, f_{n-1}(x_0)$ of $f(x)$ at x_0 exist, the nth right upper differential coefficient is defined by

$$D_n^+f(x_0)=\overline{\lim_{x\to x_0^+}}\left[f(x)-f(x_0)-\sum_{i=1}^{n-1}((x-x_0)^i/i!)f_i(x_0)\right].$$

The right lower (D_{n+}) and the left upper and lower (D_n^-, D_{n-}) nth differential coefficients are defined similarly. If the four numbers coincide they are the nth coefficient $f_n(x_0)$. Generalizing his famous results on D_1, A. Denjoy [Fund. Math. **25**, 273–326 (1935)] proved that finiteness of D_n^+ and D_{n+} imply, except for a set of measure zero, the existence of $f_n(x)$. If $D_1^+, D_{1+}, D_1^-, D_{1-}$ are all infinite, then $D_1^+=D_1^-=\infty$ and $D_{1+}=D_{1-}=-\infty$. This property does not carry over to higher derivatives. The author gives an example of a function $f(x)$ satisfying a Lipschitz condition and for which in a set of positive measure $D_2^+=D_{2+}=\infty$ and $D_2^-=D_{2-}=-\infty$. *H. Busemann* (Chicago, Ill.).

Gagaiev, B. Generalization of a Baire's theorem. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 3–5 (1943). [MF 8691]

The author considers $f(x)$ a Baire function of class $\alpha(f)$ on M, a set $N \subset M$ and a Baire function $\phi(x)$ of class $\alpha(\phi)$ on M such that $\phi(x)=f(x)$ on $M-N$. Properties of N are sought which imply $\alpha(\phi) \leqq \alpha(f)$. In case $\alpha(f)=1$, a necessary and sufficient condition that that this be so is that NK be an $F_{\alpha(f)}$ relative to M for all Borel sets K. If $N=\sum_k N_k$, where the N_k are disjoint F_γ sets, and the class $\alpha(f-\phi) \leqq \beta$ relative to M on each N_k, then $\alpha(\phi) \leqq \alpha(f)$ if $\alpha(f) \geqq 2+\max(\gamma, \beta)$ and $\alpha(\phi)=\alpha(f)$ if $\alpha(f) \geqq 3+\max(\gamma, \beta)$. *L. W. Cohen.*

Fichera, Gaetano. Intorno al passaggio al limite sotto il segno d'integrale. Portugaliae Math. **4**, 1–20 (1943). [MF 8629]

The author discusses various general problems about passage to the limit under the integral sign. In particular, he shows that uniform summability in a set E is necessary as well as sufficient for the interchange of limit and integral in the case of a sequence of summable functions converging almost everywhere in E, where E is any measurable set with finite or infinite measure. For a finite set E, uniform summability of a sequence of functions is equivalent to equi-absolute continuity of their integrals. An example is given to show that uniform summability is a more general criterion for the interchange of limit and integral than uniform boundedness by a summable function. Uniform summability is also shown to be necessary and sufficient for such an interchange in the case of a sequence which converges in measure. Moreover, such an interchange is valid uniformly for all subsets of E only if the sequence converges in measure. For convergence almost everywhere to imply convergence in measure for sets of infinite as well as finite measure, it is sufficient that the sequence be uniformly summable. A theorem of Carathéodory is proved on the summability of the superior and inferior limits of a sequence of summable functions which has a uniform bound which is summable. Various criteria are given for uniform summability. Finally, there is a discussion of the case where the range of integration as well as the integrand forms a variable sequence.
P. W. Ketchum (Urbana, Ill.).

Sanders, S. T., Jr. A correction to "A linear transformation whose variables and coefficients are sets of points." Bull. Amer. Math. Soc. **49**, 938 (1943). [MF 9696]

Cf. the same Bull. **48**, 440–447 (1942); these Rev. **4**, 74.

Theory of Functions of Complex Variables

Solovieff, P. Sur un problème limite dans la théorie des fonctions analytiques. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 191–193 (1941). [MF 9629]

The author gives the solution of the following problem: find a function $f_a(z)$ analytic outside a closed simple rectifiable curve L with $f(\infty)=A$, and a function $f_i(z)$ analytic inside L such that $f_a(z)=a(z)f_i^2(z)+b(z)f_i(z)+c(z)$ for z on L. This problem has been studied earlier by I. Privaloff [Rec. Math. [Mat. Sbornik] **41**, 527–550 (1935)]. In the present paper solutions are given in the form of a Cauchy integral in the following cases. In all cases $a(z)$, a function given on L should satisfy a Hölder condition with a positive exponent and $\ln a(z)$ should be uniform on L. Case (i): $b\equiv c\equiv 0$: here the condition of uniformity can be relaxed and the increase $\mathrm{ind}(a(z))$ of $\ln a(z)/2\pi i$ along L can be $-2n$. Case (ii): $b\equiv 0$ and $c(z)$, beside satisfying a Hölder condition, should belong to a certain wide class of functions: $\mathrm{ind}\, a(z)$ can again be $-2n$. Case (iii): $\mathrm{ind}\, a(z)=0$, $b^2=4ac$. Case (iv): $c\equiv 0$, $\mathrm{ind}\, a(z)=0$, $\mathrm{ind}\, b(z)=0$. The equivalence is pointed out between the above problem and a certain system of two integral equations [cf. Privaloff, loc. cit.] whose solutions in the above cases have thus been obtained.
František Wolf (Berkeley, Calif.).

Wishard, Audrey. Functions of bounded type. Duke Math. J. **9**, 663–676 (1942). [MF 7921]

The author determines necessary and sufficient conditions for a function $f(z)$, meromorphic in a vertical strip G, to be of bounded type in G and determines the smallest positive harmonic function $U_1(z)$ such that $\log|f(z)|\leqq U_1(z)$ in G. The method is to set up by means of Green's formula a function $U_1(z, R)$ harmonic in a rectangle interior to G having the property that $\lim_{R\to G} U_1(z, R)=U_1(z)$ is a harmonic function if and only if $f(z)$ is of bounded type in G. The condition for this turns out to be that (1) a certain sum extended over the poles of $f(z)$ in G must converge and (2) a mean value of $f(z)$, expressed as a weighted integral, on curves which approach the boundary of G, must remain finite. There are three theorems covering the cases where $f(x+iy)$ is meromorphic for (A) $0\leqq x\leqq\pi$, (B) $0<x<\pi$, (C) $x>0$, respectively. Reference is made to papers of Ahlfors, Hille and R. Nevanlinna. The paper is based on the material of the author's doctoral dissertation done under the direction of Ahlfors. *H. S. Wall* (Evaston, Ill.).

Nilson, E. N. and Walsh, J. L. Interpolation and approximation by functions analytic and bounded in a given region. Trans. Amer. Math. Soc. **55**, 53–67 (1944). [MF 9874]

Results of Walsh on interpolation and approximation by sequences of rational functions [Trans. Amer. Math. Soc. **47**, 254–292 (1940); these Rev. **1**, 309] and on approximation by bounded analytic functions [Proc. Nat. Acad. Sci. U. S. A. **24**, 477–486 (1938)] are extended, chiefly in the direction of much more general hypotheses on the regions involved, with new methods of proof. Extensions to harmonic functions are also indicated. *R. P. Boas, Jr.*

Ferrand, Jacqueline. Sur l'itération des fonctions analytiques. C. R. Acad. Sci. Paris **212**, 1068–1071 (1941). [MF 9637]

Remarks are made on a previous note [the same C. R. **212**, 977–980 (1941); these Rev. **5**, 37] on the iteration of an analytic function which carries a given simply-connected region bounded by a continuum into itself. In particular, sufficient conditions for convergence of the iterates to a unique point of the boundary and for the existence of an angular derivative with specified properties at this point are given. *M. H. Heins* (Chicago, Ill.).

Ferrand, Jacqueline. Sur les conditions d'existence d'une dérivée angulaire dans la représentation conforme. C. R. Acad. Sci. Paris **213**, 638–640 (1941). [MF 9646]

An amelioration of results of B. Grootenboer [Bull. Soc. Math. France **61**, 128–140 (1933)] on "valid" regions (domaines valables) is given by use of the theory of harmonic measure and the methods of Ahlfors.
M. H. Heins (Chicago, Ill.).

Denjoy, Arnaud. Représentation conforme des aires limitées par des continus cycliques. C. R. Acad. Sci. Paris **213**, 975–977 (1941). [MF 9664]

According to Denjoy a cyclic continuum is a continuum which contains only a finite number of disjoint continua with diameter greater than a common positive number and which is the boundary of a region. The following theorem is established. Let C denote $|z|<1$; let Γ denote $|z|=1$. If $f(z)=\sum a_nz^n$ is analytic, univalent and bounded in C, a necessary and sufficient condition that the series $\sum a_nz^n$ converge uniformly on Γ is that the image of C with respect to $f(z)$ have a cyclic continuum for a boundary. The proof is established by applying Schwarz's inequality to the condition that the area of the image of C is finite. It is further shown that, if $\sum a_nz^n$ converges uniformly on Γ with limit function $\nu(\theta)$, $f(z)$ being univalent and bounded in C, and if in addition the increasing set $h_0=0<h_1<\cdots<h_i<\cdots<h_n=2\pi$, variable with a positive number ω, satisfies the condition that $(h_i-h_{i-1})/\omega+\omega/(h_i-h_{i-1})$ is bounded, then $\sum|\nu(\theta+h_i)-\nu(\theta+h_{i-1})|^2$ converges in measure to zero when ω tends to zero. Examples illustrating the applications of this theorem are given. *M. H. Heins* (Chicago, Ill.).

Denjoy, Arnaud. Sur la représentation conforme. C. R. Acad. Sci. Paris **212**, 1071–1074 (1941). [MF 9638]

The author points out that certain results of Wolff [Nederl. Akad. Wetensch., Proc. **33**, 1185–1188 (1930)] in the theory of conformal mapping admit considerable generalization. In the present note the following theorem is established. Let $f(z)$ be defined and single-valued in the annulus C': $(\rho\leqq|z|<1)$, analytic for $|z|=\rho$, meromorphic in the interior. It is supposed that the spherical area of the Riemannian image of C' with respect to $f(z)$ is finite and that the set I of accumulation values of $f(z)$ $(|z|\to 1)$ does not coincide with the whole w-plane. Then $f(z)$ has the following properties. For almost all a of $|z|=1$, if z tends to a in a sector bisected by Oa and of angular opening less than π, then (i) $f(z)$ tends to a unique finite limit b, (ii) $(f(z)-b)(z-a)^{\frac{1}{2}}$ tends to zero, (iii) $f'(z)(z-a)^{\frac{1}{2}}$ tends to zero. If Γ is a Jordan arc possessing a tangent at each point, lying in C', abutting on $|z|=1$ at one of the aforementioned a, Γ lying in a sector of the above type, and if further the

tangent to the arc (za) of Γ at z makes an angle less than $\pi/2-\epsilon$ with the segment za $(0<\epsilon<\pi/2$, ϵ independent of $z)$, then the image of (za) is rectifiable and is $o((z-a)^{\frac{1}{2}})$. If $L(r)$ is the spherical length of the image of $|z|=r$ $(\rho<r<1)$ with respect to $f(z)$, then $L(r)(1-r)^{\frac{1}{2}}$ tends approximately to zero as r tends to one. The proof is based on the use of Schwarz's inequality applied to the area of the Riemannian image. *M. H. Heins* (Chicago, Ill.).

Denjoy, Arnaud. Sur la répresentation conforme. C. R. Acad. Sci. Paris **213**, 15–17 (1941). [MF 9151]

This note is related to the one reviewed above. The function $u=f(z)$ is assumed single-valued and meromorphic in a ring $C':\rho\leqq z<1$ and to have the properties: (a) the total area of the Riemannian image of C' on the Riemann sphere under $f(z)$ is finite; (b) the set of accumulation points of $f(z)$ in the neighborhood of $\Gamma:(|z|=1)$ does not coincide with the whole u-plane. By maximum order of contact with Γ at the point $a=e^{i\alpha}$ of a path described by z in C' and abutting on Γ at a is meant

$$\varlimsup_{z\to a}\frac{\log[1-|z|]}{\log|z-a|}-1 \qquad (z \text{ on path}).$$

The following theorem is stated to be a consequence of the previous note: At each point a of a set $E(\subset\Gamma)$ of measure 2π, $f(z)$ tends to a unique finite limit $b(a)$ when z tends to a following an arbitrary path in C' having with Γ at a contact of maximum order less than one. It is further stated that, for a set $e(\subset\Gamma)$ of measure 2π, $f(z)-b=o((z-a)^{\frac{1}{2}})$ if z tends to a in an angle bisected by Oa and having an opening less than $\pi/2$. It is shown that, if g denotes the set of points $b=b(a)$ of the boundary of the image of C' with respect to $f(z)$ which are images of $a\epsilon e$, then the area of g is finite. *M. H. Heins* (Chicago, Ill.).

Denjoy, Arnaud. Les continus frontières d'une région et la répresentation conforme. C. R. Acad. Sci. Paris **213**, 115–117 (1941). [MF 9156]

Remarks supplementary to the note of the author reviewed above are given. Constructions are indicated for regions having interesting properties in connection with the theory developed in this note. *M. H. Heins.*

Dufresnoy, Jacques. Sur les fonctions méromorphes à caractéristique bornée. C. R. Acad. Sci. Paris **213**, 393–395 (1941). [MF 9165]

This note is concerned with one of Denjoy [cf. the third review above]. It is assumed that $w=f(z)$ is meromorphic for $|z|<1$ and that the area of the Riemannian image of $|z|<1$ with respect to $f(z)$ on the Riemann sphere is finite. The supplementary condition of Denjoy stating that the set of accumulation values of $f(z)$ for z in the neighborhood of $|z|=1$ shall not cover the whole sphere is shown to be unessential. Spherical length and the spherical derivative replace Euclidean length and the usual derivative. It is shown that there exists a set E of measure 2π on $|z|=1$ such that, if z tends to an arbitrary point a of E remaining in an angle bisected by Oa and of opening less than π, then $f(z)$ tends to a unique limit b and the spherical distance $[f(z),b]$ satisfies $[f(z),b]=o((z-a)^{\frac{1}{2}})$. The condition that z remain in an angle of the type just mentioned may be replaced by the requirement that z tend to a on a path Γ having at a contact of maximum order less than one with $|z|=1$ [see the second review above]. *M. H. Heins* (Chicago, Ill.).

Yang, Ou Tchen. Surfaces de Riemann régulières de points de ramification donnés. C. R. Acad. Sci. Paris **213**, 556–558 (1941). [MF 9175]

Being given points a_i, $i=1,\cdots,n$, on a Riemann surface S_0 and integers r_i, $i=1,\cdots,n$, the author briefly sketches a method of constructing from models of S_0 a Riemann surface S (regular with respect to S_0) such that each point of S congruent to a_i is a point of ramification of r_i models of S_0, while any point of S congruent to no a_i lies on just one model of S_0. The construction is possible whenever $n\geqq3$, or when $n=2$ and $r_1=r_2$. The minimum number of models of S_0 necessary is the least common multiple of the r_i or its double, depending on whether the number of the r_i containing the greatest power of 2 is even or odd. *L. H. Loomis* (Cambridge, Mass.).

Vladimirsky, Serge. Sur la théorie de l'aile à fente. C. R. Acad. Sci. Paris **213**, 609–612 (1941). [MF 9643]

Formulas for the conformal mapping of doubly-connected regions bounded by circular slits onto canonical regions bounded by circles are given in terms of theta functions. These formulas are applied to wing theory to treat problems pertaining to the form of the profile for maximum lift or maximum stability and related questions. *M. H. Heins.*

Bochner, S. Analytic and meromorphic continuation by means of Green's formula. Ann. of Math. (2) **44**, 652–673 (1943). [MF 9404]

Two important results in the theory of analytic functions of several complex variables bear the names of Hartogs and of Poincaré and Cousin. Hartogs' theorem states that every analytic function of several complex variables can be continued from the connected boundary of a domain into its whole interior. Poincaré proved for the entire space that meromorphic functional elements whose difference is holomorphic where they overlap can be merged into one meromorphic function. Cousin extended this result to the case of a polycylindrical domain. By making use of Green's formula for general harmonic functions, the author in the present paper connects these two results, gives new proofs of them and also obtains various new results in very general versions.

We shall summarize certain of the results obtained. A small amount of preliminary terminology will be helpful. Consider some m-dimensional simplices $B_1,\cdots,B_s$ and $(m-1)$-dimensional simplices $C_1,\cdots,C_r$, each in a fixed orientation and all disjoint, and assume that each C_p is a face of one or several B_q and that each $(m-1)$-dimensional face of each B_q occurs among the C_p. Place $B=B_1+\cdots+B_s$, $C=C_1+\cdots+C_r$ and denote by $\bar{B}_q$, $\bar{B}$, etc., neighborhoods of B_q, B, etc. Take any functional elements φ_q, $q=1,\cdots,s$, each defined separately on B_q (or $\bar{B}_q$) without any interrelation. The author terms the whole set a conglomerate function on B (or $\bar{B}$) and he denotes it by φ or $\{\varphi_q\}$. If $\{\varphi_q\}$ is defined on $\bar{B}$ then the functional elements

$$\varphi^p(\xi)=\sum_{q=1}^{s}\epsilon_{pq}\varphi_q(\xi), \qquad p=1,\cdots,r,$$

give rise to a new conglomerate function this time defined on C (ϵ_{pq} is the incidence number for the face C_p in relation to B_q). This new function is called the saltus of φ on C. Now consider a differential operator

$$\Lambda f=\sum a_{\nu_1\cdots\nu_n}\frac{\partial^{\nu_1+\cdots+\nu_n}f}{\partial\xi_1^{\nu_1}\cdots\partial\xi_n^{\nu_n}}.$$

The operator is said to have the uniqueness property if every analytic solution of $\Lambda f=0$ which is defined in the complement of a bounded domain and vanishes at infinity is indentically zero.

The author proves for example the following result. (Part IV of Theorem 4) Let a conglomerate function $f(\xi)$ be harmonic in $\bar{B}$, let it be the solution of an equation Λf with uniqueness property and let the saltuses of f and of its partial derivatives of order not greater than N vanish on C. If $\bar{B}$ is the connected boundary of a bounded domain D, then f can be continued (uniquely) into all of D. This result includes Hartogs' theorem as a special case. By interpreting other results for analytic functions of several complex variables, he obtains a generalization of Cauchy's integral formula giving the value of an analytic function $f(z_1, \cdots, z_k)$ in a domain D in terms of an integral taken over the $(2k-1)$-dimensional boundary of D. [The author points out that this result is related to work of R. Fueter and E. Martinelli [Comment. Math. Helv. **12**, 75–80 (1939); **14**, 394–400 (1941); these Rev. **1**, 115; **4**, 139]. Other results obtained are the following. (Theorem 10) If D is a simplicial domain with boundary $\bar{B}$, if $\{f_\alpha\}$ is given in D and if either system

$$(1) \quad \partial f/\partial \bar{z}_\alpha = f_\alpha(z, \bar{z}), \quad \partial f_\alpha/\partial \bar{z}_\beta = \partial f_\beta/\partial \bar{z}_\alpha, \quad \alpha, \beta = 1, \cdots, k,$$

or

$$(2) \quad \partial f/\partial x_\alpha = f_\alpha(x), \quad \partial f_\alpha/\partial x_\beta = \partial f_\beta/\partial x_\alpha, \quad \alpha, \beta = 1, \cdots, n,$$

has a solution in $\bar{B}$, then there also exists a solution in D. (Theorem 11) The system (1) has a solution if the domain D is a general polycylinder. The second of these results yields essentially a new proof of the classical theorem of Poincaré and Cousin previously referred to. The closing chapter relates to meromorphic functions on the torus. Several important results are obtained including an extension of Poincaré's theorem to functions on the torus.

W. T. Martin (Syracuse, N. Y.).

Theory of Series

Soddy, F. The three infinite harmonic series and their sums (with topical reference to the Newton and Leibniz series for π). Proc. Roy. Soc. London. Ser. A. **182**, 113–129 (1943). [MF 9774]

The harmonic series, both with and without alternating signs, and their relationship with the series obtained by continuing the progression backwards to $-\infty$ are discussed. A twelve place table of sums of the harmonic series with alternating signs for various integral values of the difference and first term of the associated arithmetic progression is included. The author's argument is difficult to follow because of the formal viewpoint adopted. No apparent justification is given for the necessary rearrangements of order of these conditionally convergent series. No reference is given to Gauss, whose integral for the sum of the harmonic series with alternating signs seems to include the author's results and more [see, for instance, Bromwich, Infinite Series, 2nd ed., Macmillan, London, 1926, p. 189].

P. W. Ketchum (Urbana, Ill.).

Buck, R. Creighton. A note on subsequences. Bull. Amer. Math. Soc. **49**, 898–899 (1943). [MF 9686]

Let T be a regular matrix method of summability and let s_n be a real bounded divergent sequence. By use of the Steinhaus result that there exists a sequence of 0's and 1's not summable T, it is shown that s_n has a subsequence, having the two limit points $\liminf s_n$ and $\limsup s_n$, which is not summable T. *R. P. Agnew* (Ithaca, N. Y.).

Buck, R. Creighton and Pollard, Harry. Convergence and summability properties of subsequences. Bull. Amer. Math. Soc. **49**, 924–931 (1943). [MF 9691]

Subsequences of a given sequence $s_1, s_2, \cdots$ may be associated with points t in the interval $0<t\leqq 1$ as follows. Let $a_n=1$ or 0 according as s_n is or is not an element of the subsequence, and let t be the point with dyadic expansion $t=.a_1a_2a_3\cdots$. A set of subsequences is said to contain "almost all" subsequences if the corresponding set of points t has measure 1. If s_n is divergent, so are almost all of its subsequences. If s_n is summable to S by the Cesàro method C_1 and $\sum s_k^2/k^2<\infty$, then almost all of the subsequences are summable C_1 to S. The sequence $(-1)^n n^{\frac{1}{2}}$ is summable C_1, but almost all subsequences are nonsummable C_1. If almost all of the subsequences of s_n are summable C_1, then the sequence itself and almost all subsequences are summable C_1 to the same value. The first result on C_1 summability implies a fundamental result in the theory of probability: if a_n is the number of zeros among the first n terms of a sequence of zeros and ones and if $\lim a_n/n$ exists, then the corresponding limit exists and has the same value for almost all subsequences. *R. P. Agnew.*

Szász, Otto. On Abel and Lebesgue summability. Bull. Amer. Math. Soc. **49**, 885–893 (1943). [MF 9684]

Given a series $a_1+a_2+\cdots$, let $s_n=\sum_1^n a_\nu$, $A(r)=\sum_1^\infty a_\nu r^\nu$, $F(t)=\sum_1^\infty a_\nu((\sin \nu t)/\nu t)$. The series is said to be summable by Lebesgue's method to sum s if the series defining $F(t)$ converges in an interval $0<t<t_0$ and if $F(t)=O(1)$ as $t\rightarrow+0$. Completing his previous results [Amer. J. Math. **64**, 575–591 (1942); these Rev. **4**, 37] the author now proves: (i) if (*) $\sum_{\nu=n}^{2n} a_\nu^+=O(1)$ as $n\rightarrow\infty$ ($n^+=\max(n, 0)$) holds, then the statements $A(r)=O(1)$ as $r\uparrow 1$, $F(t)=O(1)$ as $t\downarrow 0$, $s_n=O(1)$ are equivalent; (ii) if (*) holds, then A summability implies L summability, but not necessarily convergence; (iii) if (*) holds, and if $a_1+a_2+\cdots$ converges, then $\sum a_n \sin nt/nt$ converges uniformly in $(0, \pi)$.

A. Zygmund (South Hadley, Mass.).

Bellman, Richard. Lambert summability of orthogonal series. Bull. Amer. Math. Soc. **49**, 932–934 (1943). [MF 9692]

A series $a_1+a_2+\cdots$ is said to be summable by Lambert's method to sum s if

$$\lim_{x\rightarrow 1-0} (1-x) \sum_1^\infty (na_nx^n/(1-x^n))=s.$$

The author shows that Lambert's summability of an orthonormal series $\sum a_n\phi_n(x)$ ($\sum a_n^2<+\infty$) is almost everywhere equivalent to the $(C, 1)$ summability of the series. As the author points out, this theorem is contained in known results, but the proof he gives is at certain points simpler.

A. Zygmund (South Hadley, Mass.).

Nachbin, Leopoldo. On almost everywhere divergent series of functions. Univ. Nac. Tucumán. Revista A. **3**, 311–315 (1942). (Spanish) [MF 9273]

The following theorem is proved. Let $\phi_p(x)$, $p=1, 2, 3, \cdots$, be a sequence of real functions of the real variable x, absolutely summable on a set Ω of positive measure (Lebesgue).

The divergence of the series $\sum|\lambda_p|$ (λ_p real) implies the divergence almost everywhere in Ω of the series $\sum|\lambda_p\phi_p(x)|$ if and only if

$$\liminf_{m\to\infty}\int_\Delta|\phi_m(x)|dx>0$$

for every subset Δ of Ω which has positive measure.

H. S. Wall (Evanston, Ill.).

Thron, W. J. Convergence regions for the general continued fraction. Bull. Amer. Math. Soc. 49, 913–916 (1943). [MF 9689]

The author studies convergence regions for the continued fraction

$$\frac{a_1|}{|b_1}+\frac{a_2|}{|b_2}+\cdots.$$

Application is made to

$$\frac{a_1|}{|b_1+z}+\frac{a_2|}{|b_2+z}+\cdots,$$

important in the theory of the moment problem.

J. A. Shohat (Philadelphia, Pa.).

Thron, W. J. Two families of twin convergence regions for continued fractions. Duke Math. J. 10, 677–685 (1943). [MF 9750]

The author continues the study of convergence regions for the continued fraction

$$(1)\qquad 1+\frac{c_1^2|}{|1}+\frac{c_2^2|}{|1}+\cdots.$$

Thus (1) converges if $|c_{2n-1}|\leqq s<1$, $|c_{2n}-i|\geqq s+\epsilon$, $|c_{2n}+i|\geqq s+\epsilon$, $n\geqq1$, $\epsilon>0$, arbitrarily small. Applications are made to

$$1+\frac{c_1^2x|}{|1}+\frac{c_2^2x|}{|1}+\cdots.$$

J. A. Shohat (Philadelphia, Pa.).

Fourier Series and Generalizations, Integral Transforms

Straiton, A. W. An application of Fejer summability. Nat. Math. Mag. 18, 106–107 (1943). [MF 9760]

Losinsky, S. M. On an analogy between the summation of Fourier series and that of interpolation trigonometric polynomials. C. R. (Doklady) Acad. Sci. URSS (N.S.) 39, 83–87 (1943). [MF 9719]

Let $f(x)$ be any integrable function of period 2π, and let

$$(*)\qquad \begin{aligned} s_n(f,x)&=\tfrac12a_0+\sum_{\nu=1}^{\infty}(a_\nu\cos\nu x+b_\nu\sin\nu x),\\ t_n(f,x)&=\tfrac12a_0^{(n)}+\sum_{\nu=1}^{n}(a_\nu^{(n)}\cos\nu x+b_\nu^{(n)}\sin\nu x)\end{aligned}$$

be, respectively, the nth partial sum of the Fourier series of f and the interpolating trigonometric polynomial coinciding with f at $2n+1$ equidistant points of the period. Let $t_{n,\nu}(f,x)$ denote the νth partial sum of (*). Given an infinite triangular matrix $\{k_{mn}\}$, $0\leqq m\leqq n$, $0\leqq n<\infty$, we set

$$\sigma_n(f,x)=\sum_{\nu=0}^{n}k_{n\nu}s_\nu(f,x),\qquad \tau_n(f,x)=\sum_{\nu=0}^{n}k_{n\nu}t_{n\nu}(f,x).$$

Suppose now that the matrix has the property that $\sigma_n(f,x)$ tends uniformly to $f(x)$ for every continuous f. Then, for every $f\epsilon L^p$, and for every absolutely continuous and periodic F with $F'\epsilon L^p$, $p\geqq1$,

$$\int_0^{2\pi}|f(x)-\sigma_n(f,x)|^pdx\to0,\qquad \int_0^{2\pi}|F'(x)-\tau_n'(F,x)|^pdx\to0.$$

These (and certain more general theorems) unify and extend certain results obtained previously. *A. Zygmund.*

Buchholz, Herbert. Eine einfache Reihentransformation bei einer sehr allgemeinen Fourierschen Reihe. Z. Angew. Math. Mech. 22, 277–286 (1942). [MF 8913]

The paper is concerned with the problem of summing the Fourier sine and cosine series in which the coefficient of the nth term has the form $(\pm r)^n(n-a)^{-k}$, where $0\leqq|a|<1$, r is between 0 and 1 and k is nonnegative. The cases of special interest are those for which k is small (less than about 5) and r is near 1, since direct summation of the series for these values of the parameters is difficult. The author transforms the given series into another series whose terms involve the generalized Riemann zeta function and the logarithmic derivative of the gamma function. Tables of these functions are already available, especially if k is a positive integer, in which case the zeta function is expressible in terms of Bernoulli polynomials. To obtain his series transformation the author applies an integral formula of Mellin to the terms of a Dirichlet series of appropriate form, and then equates real and imaginary parts.

P. W. Ketchum (Urbana, Ill.).

Hardy, G. H. and Rogosinski, W. W. Notes on Fourier series. II. On the Gibbs phenomenon. J. London Math. Soc. 18, 83–87 (1943). [MF 9702]

Let $s_n(\theta)$ denote the partial sums of the Fourier series of $f(\theta)$. Define $C=\frac12[f(\xi-)+f(\xi+)]$, $D=f(\xi+)-f(\xi-)$. The Gibbs set $G(\xi)$ is the set of limits of $s_n(\theta)$ when $n\to\infty$, $\theta\to\xi$ in any manner. ($G(\xi)$ is either the single point C, or a closed interval, finite or infinite.) For a large class of functions, including the classical example $\sum\sin n\theta/n$, $G(\xi)$ is a closed finite interval of length greater than D, symmetric in C. Rogosinski proved that, if f is continuous at ξ, then G is symmetric in C. In this paper a function $f(\theta)$ is constructed, continuous except for a jump at $\theta=\xi$, whose Gibbs set is finite, but not symmetric in C. *H. Pollard.*

Nikolsky, S. Sur l'évaluation asymptotique du reste dans l'approximation au moyen des sommes de Fourier. C. R. (Doklady) Acad. Sci. URSS (N.S.) 32, 386–389 (1941). [MF 9600]

Let KH^α be the class of functions $f(x)$ of period 2π and satisfying the condition $|f(x'')-f(x')|\leqq K|x''-x'|^\alpha$, where $0<\alpha\leqq1$ and $K>0$ are fixed. Let $s_n(x;f)$ be the nth partial sum of the Fourier series of f. Then $\max|f(x)-s_n(x;f)|$ for all x and all $f\epsilon KH^\alpha$ is

$$K2^{\alpha+1}\pi^{-2}n^{-\alpha}\log n\int_0^{\pi/2}t^\alpha\sin t\,dt+O(n^{-\alpha}).$$

An analogous estimate is obtained for functions f having a kth derivative belonging to KH^α. *A. Zygmund.*

Salem, R. A singularity of the Fourier series of continuous functions. Duke Math. J. 10, 711–716 (1943). [MF 9753]

There is a continuous function $f(x)$ whose Fourier series converges uniformly and yet the Fourier series of f^2 diverges at an everywhere dense set of points of the power of the continuum. *A. Zygmund* (South Hadley, Mass.).

Zygmund, A. Complex methods in the theory of Fourier series. Bull. Amer. Math. Soc. 49, 805–822 (1943). [MF 9555]

This is an address delivered before the American Mathematical Society in September, 1943 and is devoted to the application of the methods of analytic functions to the study of the trigonometric series. Although the paper does not contain more than 15 pages, it constitutes a complete guide to the researches made in this field during the past 20 or 25 years, which have led to fundamental results. Some particularly simple proofs are sketched. Special emphasis is put on the methods based on the classes of analytic functions H^p (leading to the results of F. and M. Riesz and of Hardy and Littlewood), on the method of conformal representation (theorem of Privaloff and others of the same type; theorem on uniqueness of F. Wolf) and the method of Littlewood and Paley in the study of the partial sums and of the convergence factors for functions of the class L^p. *R. Salem* (Cambridge, Mass.).

Bellman, Richard. A generalization of a Zygmund-Bernstein theorem. Duke Math. J. 10, 649–651 (1943). [MF 9747]

Let $s(t)=\sum a_k e^{i\lambda_k t}$ be a finite sum, the exponents λ_k belonging to an interval $(-\omega, \omega)$. Then

$$(*)\qquad M_p[s'(t)]\leqq \omega M_p[s(t)],\qquad 1\leqq p\leqq \infty,$$

where

$$M_p[s(t)]=\lim_{T\to\infty}\left((2T)^{-1}\int_{-T}^{T}|f(t)|^p dt\right)^{1/p}.$$

For $p=\infty$, (*) is to be interpreted as

$$\max|s'(t)|\leqq \omega \max|s(t)|.$$

A. Zygmund (South Hadley, Mass.).

Bellman, Richard. Almost periodic gap series. Duke Math. J. 10, 641–642 (1943). [MF 9745]

If $\sum_1^\infty a_k e^{i\lambda_k t}$, $\lambda_{k+1}/\lambda_k>\lambda>1$, is the Fourier expansion of a function $f(t)$ belonging to B. a. p., then

$$\lim_{T\to\infty} T^{-1}\int_0^T |f(t)|^p dt$$

exists for every $p\geqq 1$. *A. Zygmund.*

Bystrenin, V. On the approximation theorem in the theory of almost periodic functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) 33, 390–392 (1941). [MF 9550]

If $S_n(x)$ are the partial sums of a periodic Fourier series then the Bernstein expressions

$$\sigma_n(x)=\tfrac{1}{2}\{S_n(x)+S_n(x+(2\pi/(2n+1)))\}$$

have properties similar to those of Fejér sums. The author discusses the analogous situation for almost periodic series with finite basis. *S. Bochner* (Princeton, N. J.).

***Doetsch, Gustav. Theorie und Anwendung der Laplace-Transformation.** Dover Publications, N. Y., 1943. xiii+439 pp. \$3.75.

Photoprint of the (only) German edition of 1937 [J. Springer, Berlin]. A German-English glossary of some 360 scientific and technical terms has been added to the present edition.

Special Functions

Shastri, N. A. Some integrals involving products of Laguerre polynomials. Proc. Benares Math. Soc. (N.S.) 3, 23–35 (1941). [MF 9435]

The integral

$$\int_0^\infty x^{\mu-1}e^{-(\alpha+\frac{1}{2})x}W_{k,\nu}(x)\,{}_rF_s\left(\begin{matrix}l_1, & l_2, & \cdots, & l_r\\ m_1, & m_2, & \cdots, & m_s\end{matrix}; yx^p\right)dx$$

is represented as a power series in α. Here p is a nonnegative integer, W denotes Whittaker's function and ${}_rF_s$ the generalized hypergeometric function. The coefficients involve generalized hypergeometric functions of the type ${}_{r+2p}F_{s+p}$. Various special cases are considered.

G. Szegö (Stanford University, Calif.).

Mital, P. C. On integrals involving Legendre functions. Proc. Benares Math. Soc. (N.S.) 3, 17–21 (1941). [MF 9434]

The integrals, evaluated in terms of generalized hypergeometric functions, are

$$I_1=\int_0^1 P_p(1-2x^2)P_q(1-2x^2)(xz)^{-2m}\times\exp(-x^2z^2/2)M_{k,m}(x^2z^2)dx,$$

p and q positive integers, $2m$ not a negative integer,

$$I_2=\int_0^1 P_{n+\frac{1}{2}}(x)J_m(2px)x^{\mu-\frac{1}{2}}dx,\qquad \Re(n)>-\tfrac{1}{2},$$

$$I_3=\int_0^1 P_{n+\frac{1}{2}}(x)J_l(2px)J_m(2px)x^{\mu-\frac{1}{2}}dx,\qquad \Re(n)>-\tfrac{1}{2}.$$

The author lists some simpler formulas obtained by assigning special values to the various parameters.

M. C. Gray (New York, N. Y.).

Varma, R. S. An infinite series of Weber's parabolic cylinder functions. Proc. Benares Math. Soc. (N.S.) 3, 37 (1941). [MF 9436]

The formula

$$\sum_{r=0}^\infty \frac{(m+2r)(m+r-1)!\Gamma(m+2r+\frac{1}{2})}{r!}D_{-m-2r-1}(pe^{i\pi/4})\times D_{-m-2r-1}(pe^{-i\pi/4})=\frac{\pi^{\frac{1}{2}}\Gamma(2m+1)}{4^m p^{2m+1}}$$

is obtained from an integral formula for $J_m(\frac{1}{2}x^2)$ given by C. S. Maijer [Quart. J. Math., Oxford Ser. 6, 241–248 (1935)]. *M. C. Gray* (New York, N. Y.).

Banerjee, D. P. On infinite integrals containing parabolic cylinder functions. Proc. Benares Math. Soc. (N.S.) 3, 13–15 (1941). [MF 9433]

The addition theorem

$$f(x+y)=\exp[(x+y)^2/4]D_m(x+y)$$
$$=m!\exp(x^2/4)\sum_{r=0}^m(D_{m-r}(x)y^r/((m-r)!r!))$$

is used to obtain various integrals having $f(x+y)$ as part of the integrand. The reader should note that there are several misprints in the text; in particular, the last formula is meaningless as written. It should read

$$\int_{-\infty}^{\infty} \exp\left[-(y-x)^2+(y+x)^2/4\right]D_n(x+y)dy = \pi^{\frac12}2^{-n/2}e^{2x^2}D_n(2x\sqrt{2}).$$

M. C. Gray (New York, N. Y.).

Shabde, N. G. On certain relations between Bessel and Laguerre functions. Proc. Benares Math. Soc. (N.S.) **3**, 11 (1941). [MF 9432]

Two expansions are derived from the relation

$$(1+w)^a e^{-yw}=\sum_{n=0}^{\infty} L^{(a-n)}(y)w^n,$$

which is Erdélyi's extension of a result given by Deruyts in 1888. The first expansion is merely the Taylor expansion of the function

$$(z+h)^{\frac12 a}J_a[2(z+h)^{\frac12}]$$

with $z=ax$ and $h=axw$. The second expansion may be written symbolically in the form

$$(1+tE)^a J_0[2x^{\frac12}(y+1)^{\frac12}]=\sum_{n=0}^{\infty} L^{(a-n)}(y)x^{\frac12 n}J_n(2x^{\frac12}),$$

where $t=x/(y+1)$ and E is an operator which changes $J_m(z)$ into $J_{m+1}(z)$. In this expansion a is an integer.

H. Bateman (Pasadena, Calif.).

Brixy, Eduard. Zur Berechnung der Funktionen

$$J_{\nu+1}(x)/J_\nu(x) \quad \textbf{und} \quad \lg\left[J_{\nu+1}(x)/J_\nu(x)\right].$$

Z. Angew. Math. Mech. **20**, 359–361 (1940). [MF 8673]

A continuation of an earlier paper [Z. Angew. Math. Mech. **19**, 372–379 (1939); these Rev. **2**, 45]. Additional series expansions and identities involving ratios of Bessel functions, and suitable for calculation, are obtained.

P. W. Ketchum (Urbana, Ill.).

Gupta, H. C. Some infinite integrals involving Bessel functions. Bull. Calcutta Math. Soc. **35**, 7–11 (1943). [MF 9371]

The integrals in question are

$$\text{(i)} \qquad \int_0^\infty x^n J_n(xt)D_m(xe^{\frac14\pi i})D_m(xe^{-\frac14\pi i})dx,$$

which is evaluated in terms of the generalized hypergeometric series ${}_3F_3$, and (ii) two rather general integrals similar to and obtained from Gegenbauer's integral [G. N. Watson, Bessel Functions, Cambridge University Press, 1922, 13.47 equation (13)]: they contain in place of $J_\mu(bt)$ of the original integrand generalized hypergeometric series, namely, ${}_2F_2$ and a combination of ${}_3F_3$, respectively. The method of proof of (ii) is not convincing; it consists in multiplying Gegenbauer's formula, which is valid only for $b>2a>0$, by certain functions of b and then integrating with respect to b from 0 to ∞.

A. Erdélyi.

Mohan, B. Infinite integrals involving Bessel functions. II. Bull. Calcutta Math. Soc. **34**, 171–175 (1942). [MF 8998]

The integral

$$\int_0^\infty x^{\nu+1}e^{-a^2x^2}J_\nu(bx)\ {}_1F_1(\alpha;\beta;a^2x^2)dx$$

is expressed in terms of a Whittaker function. Numerous special cases involving Laguerre functions, Bessel functions, etc., arise on specializing the parameters. Misprint: Z appears in place of 2 in several places.

R. P. Boas, Jr.

Pasricha, B. R. Some integrals involving Humbert function. Proc. Indian Acad. Sci., Sect. A. **18**, 11–19 (1943). [MF 9062]

The author computes various integrals involving Humbert's function

$$J_{m,n}(x)=(x/3)^{m+n}\{\Gamma(m+1)\Gamma(n+1)\}^{-1} \times {}_0F_2(m+1, n+1; -x^3/27).$$

The method used is that of the operational calculus.

G. Szegö (Stanford University, Calif.).

Pasricha, B. R. Some infinite integrals involving Whittaker functions. J. Indian Math. Soc. (N.S.) **7**, 46–50 (1943). [MF 9381]

Various integrals are considered involving Whittaker's functions, in particular, Laguerre's polynomials and Bessel's functions.

G. Szegö.

Tsvetkoff, G. E. Sur les racines complexes des fonctions de Whittaker. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 290–291 (1941). [MF 9573]

The author states, without proof, some theorems on the location of the complex roots of the functions $M_{k,m}(z)$ and $W_{k,m}(z)$ for real values of k and m similar to the theorems on the real roots given in an earlier paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 10–12 (1941); these Rev. **3**, 237].

M. C. Gray (New York, N. Y.).

Differential Equations

Bystrenin, V. On almost periodic solutions of some ordinary differential equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 387–389 (1941). [MF 9549]

The author sets up formulas for solutions of a system of differential equations

$$dy_i/dx=\sum_j f_{ij}(x)y_j(x)+G_i(x)$$

with a view to deciding when such solutions are almost periodic.

S. Bochner (Princeton, N. J.).

Schneider, Peter. Beitrag zur vollständigen Lösung der Differentialgleichung freier gekoppelter Schwingungen für beliebige Dämpfung sowie Art und Grösse der Kopplung. Ann. Physik (5) **41**, 211–224 (1942). [MF 9231]

This paper is an extension of the work of K. W. Wagner on the characteristic frequencies of oscillating electrical systems with limited coupling and damping. Based on the above the author solves and discusses the differential equations of such oscillations under general coupling and damping.

A. L. Foster (Berkeley, Calif.).

Bulgakov, B. V. On the problem of forced vibrations of pseudo-linear systems. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 31–40 (1943). (Russian. English summary) [MF 9723]

The author considers a pseudo-linear conservative system of the form $m\ddot{x}+f(x)=R\sin wt$. Applying a frequently used approximation procedure, popularized by van der Pol, a

simplified pair of equations are obtained and solved. Information concerning stability and the occurrence of beats is obtained from the energy integral. The possible types of behavior of the system are summed up diagrammatically.
N. Levinson (Cambridge, Mass.).

Bulgakov, B. V. Maintained oscillations of automatically controlled systems. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 97–108 (1943). (Russian. English summary). [MF 9730]

The author gives the details of a paper which appeared earlier [C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 250–253 (1942); these Rev. **4**, 275]. *N. Levinson.*

Sommerfeld, A. Die frei schwingende Kolbenmembran. Ann. Physik (5) **42**, 389–420 (1943). [MF 9240]

The mathematical interest is in the solution of (1) $\nabla^2\varphi+k^2\varphi=0$ subject to (2) $(\partial\varphi/\partial z)|_{z=0}=1$, $r<a$, (3) $(\partial\varphi/\partial z)|_{z=0}=0$, $r>a$. Write

$$\phi(r,z)=\int_0^\infty e^{-\mu z}J_0(\lambda r)f(\lambda)d\lambda, \qquad \mu^2=\lambda^2-k^2, \tag{4}$$

and the problem amounts to a determination of $f(\lambda)$ so that (2) and (3) are satisfied. Assume

$$f(\lambda)=\left(\sum_{\sigma=1}^{\infty}A_\sigma J_{\sigma+\frac{1}{2}}(\lambda a)/(\lambda a)^\sigma\right)(\pi\lambda a/2)^{\frac{1}{2}}. \tag{5}$$

From (4) and (5) it follows that (1) and (2) are satisfied, and the problem is reduced to finding A_σ consistent with (3). The practical determination of A_σ is a formal procedure of great beauty by reason of a method due to Perron. Write (4) as $R+iI$. Suppose $I\equiv 0$; then A_σ is real and there arises, after extensive manipulation of Bessel's functions, the infinite system

$$\sum_{1}^{\infty}A_\sigma J_{\sigma-n}(ka)/(ka)^{\sigma-n}=C_n, \quad n=0,1,2,\cdots, \tag{6}$$

where C_n is a constant given in terms of k and a. A classical Laurent's expansion is

$$e^{-ka/2t}=e^{-kat/2}\sum_{-\infty}^{\infty}J_n(ka)t^n. \tag{7}$$

(Sommerfeld uses arguments about plane waves for this.) Expand $e^{-ka/2t}$ and $e^{-kat/2}$ in powers of $1/t$ and t, respectively, and equate coefficients of t on both sides of (7). The resulting system of identities is denoted by (8). Multiply the mth equation in (6) by $[(-1)^{n-m}/(n-m)!2^{n-m}]1(n-m)$, where $1(t)=1$ for $t\geqq 0$ and is 0 for $t<0$. Sum on n. Then in view of (8) only the coefficients of $A_1,\cdots,A_m$ differ from 0 and the right side is an infinite series (in C_n) which we denote by d_n. In short, (6) is replaced by a system (9) whose matrix of coefficients has zeros to the right of the diagonal. To solve (9) multiply the mth equation by z^m and sum on m. It develops that the left side is $\sum_1^\infty A_\sigma z^{\sigma+1}e^{\alpha z}$, $\alpha=(ka)^2$. Hence we can solve explicitly for A_σ by comparing coefficients of z^n in $\sum A_\sigma z^{\sigma+1}=(\sum d_n z^n)e^{-\alpha z}$. Unfortunately Perron's elegant solution cannot be maintained when $I\not\equiv 0$. The A_σ's are now complex numbers. A perturbation method is applied to give the correction to A_σ up to terms of order $(r/a)^n$ when I is not neglected. *D. G. Bourgin.*

Ott, H. Reflexion und Brechung von Kugelwellen; Effekte 2. Ordnung. Ann. Physik (5) **41**, 443–466 (1942). [MF 9235]

The author suggests the name "Flankenwellen" for the remarkable subsidiary wave found by O. v. Schmidt with the aid of photography. Following up the idea of Joos and Teltow that an explanation of the new wave is to be found in the work of Sommerfeld, Ott considers a source L of sound waves or electromagnetic waves which is at a distance h above the reflecting surface instead of being in the reflecting surface itself. The velocity potential Φ_0 or the z-component Π_0 of the Hertzian vector in the primary wave is resolved into a family of plane waves in a manner indicated by Weyl:

$$\Pi_0=(1/R)e^{-ikR}$$
$$=(ik/2\pi)\int\int e^{ik[r\sin\theta\cos(\phi-\phi')+|z-h|\cos\theta]}\sin\theta d\theta d\phi.$$

The integration with respect to θ is from 0 to $\frac{1}{2}\pi-i\infty$; that with respect to ϕ is from 0 to 2π and can be carried out, giving $2\pi J_0(kr\sin\theta)=\pi H_0^1(\sin\theta)+\pi H_0^2(\sin\theta)$ according to Sommerfeld's plan of splitting up the wave. In the second partial integral θ is now replaced by $-\theta$ and use is made of the relation $H_0^2(-x)=-H_0^1(x)$, giving the final form

$$\Pi_0=(ik/2)\int\int_{-\pi/2+i\infty}^{\pi/2-i\infty}H_0^1(rk\sin\theta)e^{ik|z-h|\cos\theta}$$
$$\times\sin\theta d\theta=(ik/2)\int_C.$$

For the reflected wave the corresponding form is

$$\Pi_r=(ik/2)\int_C H_0^1(rk\sin\theta)e^{ik(z+h)\cos\theta}f\sin\theta d\theta, \quad z>0,$$

where $f[\gamma\cos\theta+N]=\gamma\cos\theta-N$, $N=(n^2-\sin^2\theta)^{\frac{1}{2}}$, $\gamma=n^2$ in the optical case but $\gamma=\rho_1/\rho_2$ in the acoustical case. The transmitted wave is likewise given by

$$\Pi_g=(ik/2)\int_C H_0^1(rk\sin\theta)e^{-ikNz+ikh\cos\theta}g\sin\theta d\theta, \quad z<0,$$

where $g[\gamma\cos\theta+N]=2\cos\theta$. A lengthy discussion of these integrals follows and it is shown in particular that the "Flankenwelle" exists only in the angular space between the boundary and a cone of rays starting from the image L' of L at an angle α_0 with the axis of z, α_0 being the limiting angle of total reflection. For the case of elastic waves and sound waves the problem of reflexion and refraction has been treated with the aid of Weyl's integral by T. Sakai [Geophys. Mag. **8**, 1–71, 205–218 (1934), the second paper on sound being in collaboration with S. Syôno].
H. Bateman.

Marx, Helmut. Zur Theorie der Zylinder- und Kugelwellen in reibungsfreien Gasen und Flüssigkeiten. Ann. Physik (5) **41**, 61–88 (1942). [MF 9230]

Bechert [Ann. Physik (5) **39**, 169–202 (1941); these Rev. **3**, 282] derived the differential equation

$$\frac{dv}{dw}=\frac{(k-1)(2w+k(n-1)wv)}{k(w-1)(v(w-1)^{2(k-1)/k}-w^2)} \tag{1}$$

for the propagation of cylindrical ($k=2$) or spherical ($k=3$) waves in a viscosity free medium with state equation $p=p_0+a^2\rho^n/n$. The writer studies in great detail the integral curves, singularities and qualitative character of the solutions of (1) for a number of values of n. *D. G. Bourgin.*

Pfriem, H. Zur gegenseitigen Überlagerung ungedämpfter ebener Gaswellen grosser Schwingungsweite. Akustische Z. **7**, 56–65 (1942). [MF 9411]

This article is expository and attempts to make the classical analytical treatment of air waves of finite ampli-

tude clear using physical considerations. A similar exposition is to be found in a paper by Rayleigh ["Aerial Plane Waves of Finite Amplitude"]. Many of the formulas presented in this paper are to be found in Hadamard's "Leçons sur la propagation des ondes" [Hermann et Cie, Paris, 1903]. The modern theories which involve discontinuities (shock waves, etc.) are not treated. *C. B. Morrey, Jr.*

Riabouchinsky, Dimitri. Commentaires sur la théorie des ondes planes. C. R. Acad. Sci. Paris **213**, 469–472 (1941). [MF 9170]

For one dimensional supersonic flow Riemann and Lamb have demonstrated the compulsory deformation of waves and the appearance of shock waves. In previous publications [mostly in periodicals not available to the reviewer] the author has developed a theory permitting practically invariable waves. His announced main object is the explanation of this "profounde divergence" of conclusions. Although no explicit resumé of the author's theory is presented, it appears from his comments and equations that it amounts formally to replacing the density ρ by a longitudinal (ρ_x) and a transverse (ρ_y) density. Thus one dimensional hydrodynamics can be considered as a case of two dimensional flow for which $\rho_y=\infty$ or $\epsilon=\rho_x/\rho_y=0$. Actually the author introduces ϵ alone in the paper. The resulting two dimensional equations with $\epsilon\neq 1$ are of course non-Eulerian. Lamb's solution involves the assumption $(d/dt)v_y=0$ and is consistent with the non-Eulerian equations, $\epsilon=0$, in a sort of artificial way for special flows. Riemann's solution is based on $v_y=0$ and is inconsistent with the non-Eulerian equations (apparently because of the curl equation). The author repeatedly cites experimental support of his theory which corresponds to the case $0<\epsilon\ll 1$. However he is really comparing a two dimensional flow theory (since $\epsilon>0$) with an idealized one dimensional theory. It seems to the reviewer that it would be much more to the point to compare the two dimensional analogue of Riemann's theory or rather, since no complete extension of Riemann's solution has been developed as yet, some suitable known approximant. On the other hand unless some a priori physical justification (say thermodynamical or hydrodynamical, etc.) is given for introducing ρ_x, ρ_y or ϵ, the author's "theory" seems merely an interesting formal method for introducing a new parameter, or embedding the hydrodynamical equations in a one parameter family of equations, rather than an explanation of fluid phenomena. *D. G. Bourgin.*

Piskounov, N. S. The solution of an equation in the boundary layer theory by the method of finite differences. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 157–159 (1942). [MF 8656]

Previously [same C. R. (N.S.) **27**, 104–106 (1940); these Rev. **2**, 203] the author has given proofs of the existence of the solution of the von Mises boundary layer equation $\partial^2 z/\partial y^2=[\theta(x)-z]^{-\frac{1}{2}}\partial z/\partial x$, $z(x,0)=\theta(x)$, $z(0,y)=\phi(y)$, $z(x,\infty)=0$, under the assumption that $\theta(x)$ is nonincreasing. In the present note the author gives a new existence proof based on consideration of the difference equation

$$[z(x+h,y)-z(x,y)]/h=[\theta(x)-z(x,y)]^{\frac{1}{2}} \times[z(x,y+l)-2z(x,y)+z(x,y-l)]/l^2.$$

The reviewer remarks that the above difference equation differs but slightly from that considered by Luckert [Schriften Math. Sem. und Inst. Angew. Math. Univ. Berlin **1**, 246–274 (1933)]. The author announces the following results. Let $\phi(y)$, $\theta(x)$ be continuous and bounded functions, let $\phi(0)=\theta(0)$, $\phi(\infty)=0$, $\phi(0)>\phi(y)$, $\phi(y)<\phi(0)-xy$ for $y<\epsilon$ and let $\theta(x)$ be a nondecreasing function. Then the sequence of solutions of the difference equations which satisfy the corresponding boundary conditions converges to a limit function. This limit function is the solution of the von Mises equation. If the conditions concerning ϕ and θ are dropped but if $\theta(x)$ is assumed to be a nonincreasing function and if the boundary layer equation has a solution then the sequence of the solutions of the difference equations converges to the solution of the differential equation. *S. Bergman* (Providence, R. I.).

Lyubov, B. J. Designing of unstationary temperature fields in bodies of the simplest shape. II. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **13**, 42–49 (1943). (Russian) [MF 9026]

Continuing considerations of the first part [Lyubov and Finkelstein, same Zhurnal **13**, 35–41 (1943); these Rev. **5**, 69], the author determines the temperature distribution in an infinitely long cylinder under the boundary conditions $T(R,t)$, where R is the radius of basis of the cylinder. *S. Bergman* (Providence, R. I.).

Bothe, W. Einige Diffusionsprobleme. Z. Phys. **118**, 401–408 (1941). [MF 9564]

This paper is devoted to the consideration of the integro-differential equation

$$\cos\vartheta\,\partial K(\vartheta,\xi)/\partial\xi=-K(\vartheta,\xi)+\kappa\rho(\xi), \tag{1}$$

where

$$\rho(\xi)=\tfrac{1}{2}\int_0^\pi K(\vartheta,\xi)\sin\vartheta d\vartheta \tag{2}$$

and κ is a constant. Expanding K as a series in spherical harmonics, it is first shown that ρ satisfies the equation

$$(d^2/d\xi^2)(\rho\,\overline{\cos^2\vartheta})=(1-\kappa)\rho,$$

where

$$\overline{\cos^2\vartheta}=\int_0^\pi K\cos^2\vartheta\sin\vartheta d\vartheta\Big/\int_0^\pi K\sin\vartheta d\vartheta,$$

and that accordingly a reduction to a problem in diffusion is possible only when $\overline{\cos^2\vartheta}$ is independent of ξ. Next, starting with the formal solution

$$K=\kappa\int_{\xi_0(\vartheta)}^{\xi}\rho(\zeta)e^{-(\xi-\zeta)\sec\vartheta}\sec\vartheta d\vartheta+F(\vartheta)e^{-\xi\sec\vartheta}$$

of equation (1) (where $F(\vartheta)$ governs the variation K with ϑ on a certain curve $\xi=\xi_0(\vartheta)$), it is shown how an integral equation for ρ can be derived. For the case $\xi_0(\vartheta)=-\infty$ for $0\leqq\vartheta<\pi/2$ and $\xi_0(\vartheta)=-\infty$ for $\pi/2\leqq\vartheta<\pi$, the equation is

$$\rho(\xi)=\tfrac{1}{2}\kappa\int_{-\infty}^{+\infty}d\zeta\rho(\zeta)\int_{|\zeta-\xi|}^{\infty}(dt/t)e^{-t},$$

and it is pointed out that a solution of this equation of the form $\rho(\xi)=\text{const.}\,e^{-\alpha\xi}$ exists if α is chosen as a root of the equation

$$\tfrac{1}{2}\kappa\log((1+\alpha)/(1-\alpha))=1.$$

Some further consequences of this result are discussed. [The reviewer may be permitted to add that most of the results contained in this paper have long been familiar to astro-physicists in their studies on the radiative equilibrium of a stellar atmosphere; see, for example, E. Hopf, Mathematical Problems of Radiative Equilibrium, Cambridge Tracts in Mathematics and Mathematical Physics, no. 31, Cambridge, England, 1934.] *S. Chandrasekhar.*

Bothe, W. Die Diffusion von einer Punktquelle aus. (Nachtrag zu der Arbeit "Einige Diffusionsprobleme.") Z. Phys. **119**, 493–497 (1942). [MF 9449]

In this paper the considerations of the paper reviewed above are extended to the study of the equation

$$\cos\vartheta\frac{\partial K(\vartheta,\xi)}{\partial\xi}-\frac{\sin\vartheta}{\xi}\frac{\partial K}{\partial\vartheta}=-K(\vartheta,\xi)+\kappa\rho(\xi),$$

where κ is a constant and $\rho(\xi)$ is defined as in equation (2) of the preceding review. Again starting with the formal solution of this equation, which can be readily written down, an integral equation for ρ can be derived. For the case of a point source, the appropriate equation is

$$f(\xi)=\tfrac{1}{2}\kappa\int_0^\infty d\zeta f(\zeta)\int_{|\xi-\zeta|}^{\xi+\zeta}(dt/t)e^{-t}+(Q/\xi)e^{-\xi}$$
$$=f_1+f_0$$

(say), where $f=\xi\rho$ and Q denotes the strength of the source at $\xi=0$. Expressing f and f_0 as Fourier integrals of the form

$$f(\xi)=\int_0^\infty A(\alpha)\sin\alpha\xi d\alpha,\quad f_0(\xi)=\int_0^\infty A_0(\alpha)\sin\alpha\xi d\alpha,$$

it is shown that

$$A_0(\alpha)=\frac{2}{\pi}Q\tan^{-1}\alpha,\quad A(\alpha)=\frac{2\alpha}{\pi}Q\frac{\tan^{-1}\alpha}{\alpha-\kappa\tan^{-1}\alpha}.$$

Accordingly, for this case the solution for ρ can be explicitly written down in the form

$$\rho=\frac{Q}{\xi^2}e^{-\xi}+\frac{2\kappa}{\pi}Q\frac{1}{\xi}\int_0^\infty\frac{(\tan^{-1}\alpha)^2}{\alpha-\kappa\tan^{-1}\alpha}\sin(\alpha\xi)d\alpha.$$

The definite integral occurring in this solution has been evaluated numerically for various values of ξ and graphically illustrated. *S. Chandrasekhar* (Williams Bay, Wis.).

Fortet, Robert. Sur la résolution des équations paraboliques linéaires. C. R. Acad. Sci. Paris **213**, 553–556 (1941). [MF 9174]

The differential equation $u_t+a(t,x)u_{xx}+b(t,x)u_x=0$ is closely connected with the theory of stochastic processes, and this theory leads to the investigation of a new type of properties of the solution and of relations between the coefficients and the solution. In two notes [same C. R. **212**, 325–326, 1118–1120 (1941); these Rev. **3**, 4; **5**, 125] the author has obtained many results in this direction. Here, he restates some results in the language of differential equations. They may be regarded as a generalization of Petrowski's results [Compositio Math. **1**, 383–419 (1935)]. *W. Feller* (Providence, R. I.).

Biben, Georges. A propos du principe de É. Picard. C. R. Acad. Sci. Paris **215**, 12–13 (1942). [MF 9475]
Biben, Georges. Sur une extension du principe de É. Picard. C. R. Acad. Sci. Paris **214**, 989–991 (1942). [MF 9472]

In these notes the author discusses the uniqueness of the solution of the linear partial differential equation in three variables

$$\Delta\psi+2\delta\frac{\partial\psi}{\partial x}+2\epsilon\frac{\partial\psi}{\partial y}+2f\frac{\partial\psi}{\partial z}+g\psi=0,$$

where boundary values are assigned on the boundary of a three dimensional region. It is proved by means of the usual Green formula type of integral identities that the solution is unique if the expression

$$\theta=\frac{\partial\delta}{\partial x}+\frac{\partial\epsilon}{\partial y}+\frac{\partial f}{\partial z}-g$$

is nonnegative. If θ is negative, uniqueness is proved only if the region is sufficiently small, the size depending on the minimum value of θ. *J. W. Green* (Aberdeen, Md.).

Meyman, N. Sur un problème limite pour les équations polyharmonics. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 275–278 (1941). [MF 9569]

In an n-dimensional space let D be a finite domain, bounded by S, a hypersurface of $n-1$ dimensions. Let $\varphi_\mu(P)$, $\mu=1, 2, 3, \cdots, m$, be m functions defined at each point P of S. Certain restrictions are imposed on S and on these functions, including the requirement that $\varphi_\mu(P)$ should have continuous partial derivatives of orders 1, 2, 3, $\cdots$, $m-\mu$. The principal theorem of the paper asserts that there exists a function u which is m-harmonic in D, that is, $\Delta^m u(x_1, x_2, \cdots, x_n)=0$ in D, where

$$\Delta^m u=\sum_{\Sigma\alpha=m}\frac{m!}{\alpha_1!\alpha_2!\cdots\alpha_n!}\frac{\partial^{2m}u}{\partial x_1^{2\alpha_1}\partial x_2^{2\alpha_2}\cdots\partial x_n^{2\alpha_n}},$$

and such that when the interior point Q of D tends toward the point P of S the function u and its first $m-1$ normal derivatives tend toward $\varphi_1(P)$, $\varphi_2(P)$, $\cdots$, $\varphi_m(P)$, respectively; by imposing certain further requirements the function u may be uniquely determined. *F. W. Perkins.*

Reade, Maxwell. On subharmonic functions. Bull. Amer. Math. Soc. **49**, 894–897 (1943). [MF 9685]

The following notations are needed in the sequel. If $g(x, y)$ is continuous in a domain D, then $A(g; x, y; r)$ designates the integral mean of g over the area of the circular disc with center (x, y) and radius r, while $G(g; x, y; r)$ designates the integral mean of g over the perimeter of the disc. The continuous function $g(x, y)$ is of class PG in D if (1) $g\geqq 0$ in D and (2) $\log g$ is subharmonic at every point of D where $g>0$. The author presents the following generalizations of previous results. (Theorem) Let $f(t)$ have a continuous second derivative and a positive first derivative for $-\infty<t<\infty$. If $v(x, y)$ has continuous partial derivatives of the second order in a domain D, and if

$$f\{v(x,y)+\log[(x-a)^2+(y-b)^2]\}$$

is subharmonic in D for every choice of the real constants a, b, then v is subharmonic in D. (Theorem) If $p(x, y)\geqq 0$ has continuous partial derivatives of the second order in D, and if a, b are real constants such that $(1/a)+(1/b)=2$, then p is of class PG in D if and only if the inequality

$$A(pq; x, y; r)\leqq[G(p^a; x, y; r)]^{1/a}[G(q^b; x, y; r)]^{1/b}$$

holds for every function $q(x, y)$ of class PG and for every circular disc in D. It would be clearly a matter of interest to weaken the differentiability conditions in the above theorems, but the author remarks that the usual smoothing process does not seem to work here. *T. Radó.*

Lelong, Pierre. Définition des fonctions plurisousharmoniques. C. R. Acad. Sci. Paris **215**, 398–400 (1942). [MF 9506]

A generalization of the definition of a subharmonic function to the case of several variables is given: a function V of n complex variables z_k belongs to the class of functions P in a domain D if (a) at each point of D it has a finite value

or is $-\infty$, is finite at least at one point of D and has a finite upper bound in every domain interior to D, (b) by the substitution $z_k = a_k t$, V becomes a subharmonic function in the t-plane. The average value over the surface of $|z_k| = r_k$, $k = 1, \cdots, n$, is used to establish by induction that a function of P is semicontinuous from above and that the average is a finite nondecreasing function of r_k. Finally, an equivalent definition of the class P is given: the function should be either finite or $-\infty$ at every point of D, upper semicontinuous and the above average should not be less than the value of the function at the center for any r_k.

František Wolf (Berkeley, Calif.).

Lelong, Pierre. Sur les suites de fonctions plurisousharmoniques. C. R. Acad. Sci. Paris **215**, 454–456 (1942). [MF 9510]

The author is concerned with the limits of sequences of functions belonging to P defined in the preceding review. It is pointed out that, although the limiting function of a nonincreasing, or of a uniformly convergent, sequence of functions of P is in P, P is not a closed set of functions. A wider closed class $\bar{P}^*$ of functions (fonctions presque plurisousharmonique) is defined: a function of $\bar{P}^*$ is such that it is integrable over any measurable subset of D and its average value in any sphere contained in D is not smaller than its value at the center. The necessary and sufficient conditions for a continuous function v with two continuous derivatives to belong to P are the following inequalities between its second partial derivatives: $v_{kk} \geqq 0$ and $v_{kk}v_{ll} - (v_{kl})^2 \geqq 0$. We call the class of such functions C. Hence, P consists of the functions of C plus the limits of all the nonincreasing sequences of functions of C.

František Wolf (Berkeley, Calif.).

Theory of Probability

Greville, T. N. E. Regularity of label-sequences under configuration transformations. Trans. Amer. Math. Soc. **54**, 403–413 (1943). [MF 9518]

Configuration transformations enable one to test the reasoning underlying a large variety of probability calculations. Thus corresponding to a set of observations there is assigned a probability which is computed in terms of the distribution functions associated with the various observations. If the computed probability is correct, it should be possible to exhibit it as the limit of a success ratio. More precisely, let $x_1, x_2, \cdots$ be a finite or denumerable set of sequences such that each element of each sequence x_i is an observation which may be regarded as a point of an abstract space π_i. The first r elements of each of the sequences constitute a configuration C_r. To such a configuration there may or may not be assigned a point $f(C_r)$ of a space ρ. The sequence $C_1, C_2, \cdots$ determines a sequence y (the configuration transformation of $x_1, x_2, \cdots$) consisting of those points $f(C_r)$ for which the function is defined. The distributions of the sequences x_i induce a distribution for the sequence y. If the induced or computed distribution is exhibited (except for measure zero) by the appropriate limits of success ratios, the computation is regarded as satisfactory. The author obtains rather liberal conditions under which satisfactory computations will be obtained.

A. H. Copeland (Ann Arbor, Mich.).

Hadwiger, H. Über gleichwahrscheinliche Aufteilungen. Z. Angew. Math. Mech. **22**, 226–232 (1943). [MF 8988]

Despite its title, this paper is actually concerned with the following two questions on geometric probability considered by Laplace. Let $n-1$ points, taken at random from the unit interval [0, 1], divide it into n subintervals. What is the probability that no subinterval exceed a given number $\xi < 1$? What is the expected value of the largest subinterval? The answer to the first question is simply

$$\sum_{\lambda=0}^{[\xi^{-1}]} (-1)^\lambda \binom{n}{\lambda} (1-\lambda\xi)^{n-1},$$

but is given in the present paper in a nearly unrecognizable form by means of an infinite integral. The answer to the second question is

$$(1+\tfrac{1}{2}+\cdots+1/n)/n.$$

This is a special case of Laplace's rule [see, for example, Encyclopaedia Britannica, 13th ed., v. 22, p. 384 a] which asserts that the expected value of the rth largest interval is

$$(1/r+1/(r+1)+\cdots+1/n)/n.$$

The questions are stated in terms of combinatorial Diophantine analysis as follows. Out of all the $\binom{n+s-1}{s}$ compositions of the integer s into n parts not less than 0, how many have no part exceeding ξs? What is the average size of the largest part? These questions have answers which are numerical functions of the integer s. The above results are, of course, only the leading, or "volume," terms in the asymptotic formulas for these two lattice point problems.

D. H. Lehmer (Berkeley, Calif.).

Gontcharoff, W. Sur la succession des événements dans une série d'épreuves indépendantes répondant au schème de Bernoulli. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 283–285 (1943). [MF 9576]

Consider a simple alternative A or B, with constant probabilities p and q. Denote by

$$p\{\alpha_1, \alpha_2, \cdots; \beta_1, \beta_2, \cdots\}$$

the probability that in n trials there will be α_i runs of A's and β_i runs of B's of length i each ($\sum_i \alpha_i + \sum_i \beta_i = n$). The corresponding generating function is

$$-\{1+\sum p^r x_r\}\{1+\sum q^r y_r\}\{1-\sum p^r x_r \sum q^r y_r\}^{-1}.$$

From it the author computes the expectation and variances of (1) the number of A runs, (2) the number of A and B runs, (3) the number of A and B runs of length r. Finally the distribution function of the length of the longest A run is given. The note contains no proofs.

W. Feller.

Dugué, Daniel. Sur certaines composantes des lois de Cauchy. C. R. Acad. Sci. Paris **213**, 718–719 (1941). [MF 9653]

It is shown that the Cauchy law (with the density function $\{\pi(1+x^2)\}^{-1}$ and the characteristic function $e^{-|t|}$) can be decomposed into two laws of different types. The result is interesting in view of the theorems of Cramér and Raikoff to the effect that neither the normal nor the Poisson law afford such decompositions. The author proves his observation in an utterly simple way using Pólya's theorem, according to which any even function $\phi(t)$ for which $\phi'(t) < 0$ and $\phi''(t) > 0$ for $t > 0$ is a characteristic function of a distribution function.

W. Feller (Providence, R. I.).

Mann, H. B. and Wald, A. On stochastic limit and order relationships. Ann. Math. Statistics **14**, 217–226 (1943). [MF 9139]

The authors have collected in this paper definitions and theorems concerning convergence in probability (stochastic convergence) which are useful in statistical work. The following theorem examplifies the kind of results obtained. Let $\{x_n\}$ be a sequence of r-dimensional random vectors such that the distribution function of x_n converges to the distribution function F of a random vector x at each continuity point of F. Let $g(\xi)$ (ξ an r-dimensional vector) be a Borel measurable function whose set R of points of discontinuity is closed and such that Prob $\{x\epsilon R\}=0$. Then the distribution function of $g(x_n)$ converges to the distribution function of $g(x)$ at each continuity point of the latter. Applications to finding limiting distributions are discussed briefly. *M. Kac* (Ithaca, N. Y.).

Feller, W. Generalization of a probability limit theorem of Cramér. Trans. Amer. Math. Soc. **54**, 361–372 (1943). [MF 9516]

Let $\{X_k\}$ be a sequence of individually bounded independent chance variables with mean 0 and variance σ_k^2, and let $S_n=X_1+\cdots+X_n$, $s_n^2=\sigma_1^2+\cdots+\sigma_n^2$. The author obtains asymptotic estimates for

$$1-F_n(xs_n)=\text{Prob }\{S_n\geqq xs_n\}$$

as $x, n\to\infty$. His results generalize those of Cramér [Actualités Scientifiques, no. 736, Hermann et Cie., Paris, 1939 pp. 5–23], who has considered the case where all X_k have the same distribution. A typical result is the following: if $\{X_k\}$ is uniformly bounded, then, for $x=o(n^{1/6})$, $1-F_n(xs_n)$ is of the order of magnitude of $1-\varphi(x)$, where

$$\varphi(x)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x}\exp-(\tfrac{1}{2})y^2dy;$$

for $x=o(n^{1/4})$, $1-F_n(xs_n)$ is of the order of magnitude of

$$[1-\varphi(x)]\exp(-(\tfrac{1}{2})q_{n,1}x),$$

where $q_{n,1}$ depends only on the third moments of $\{X_k\}$; for more rapidly increasing x, higher moments of $\{X_k\}$ also become significant but, for all $x=o(n^{\frac{1}{2}})$, $\log(1-F_n(xs_n))$ is still of the same order of magnitude as $\log(1-\varphi(x))$. The results are to be applied [see the following review] to an investigation of the law of the iterated logarithm.
D. Blackwell (Atlanta, Ga.).

Feller, W. The general form of the so-called law of the iterated logarithm. Trans. Amer. Math. Soc. **54**, 373–402 (1943). [MF 9517]

This paper continues Kolmogoroff's well-known paper on the iterated logarithm [Math. Ann. **101**, 126–135 (1929)]. Using the terminology of P. Lévy and the notation of the preceding review, an increasing sequence of numbers $\{\varphi_n\}$ is said to belong to the lower class $\mathfrak{L}$ if with probability 1 the inequality $S_n\geqq s_n\varphi_n$ is satisfied infinitely often, and is said to belong to the upper class $\mathfrak{U}$ otherwise. Using the estimates for $1-F_n(xs_n)$ mentioned in the preceding review the author obtains, under the hypothesis

$$(1)\qquad \left|\frac{X_k}{s_n}\right|<\lambda_n\leqq\frac{1}{200\varphi_n},\qquad k=1,2,\cdots,n,$$

necessary and sufficient conditions, in several forms, that a sequence $\{\varphi_n\}\epsilon\mathfrak{L}$. A special case of his results, generalizing a theorem of Kolmogoroff [proved by Erdös, Ann. of Math. (2) **43**, 419–436 (1942); these Rev. **4**, 16], is the following: if $\varphi(t)\uparrow$ and $\lambda_n=O(1/\varphi^3(s_n^2))$, then $\varphi(s_n^2)\epsilon\mathfrak{U}$ if and only if

$$\int^{\infty}\frac{1}{t}\varphi(t)\exp(-\tfrac{1}{2}\varphi^2(t))dt$$

converges. In particular,

$$\varphi_n=\{2\log_2 S_n^2+3\log_3 S_n^2+\cdots+2\log_{p-1}S_n^2 +(2+\delta)\log_p S_n^2\}^{\frac{1}{2}}\epsilon\mathfrak{U}(\mathfrak{L})$$

if $\delta>0(<0)$. The following sharpening of the notion of lower class is obtained: under the hypothesis (1) if $\{\varphi_n\}\epsilon\mathfrak{L}$, then, for every constant a, $\{\varphi_n+a/\varphi_n\}\epsilon\mathfrak{L}$. By the method of equivalent sequences, results are also obtained for unbounded sequences $\{X_n\}$. *D. Blackwell* (Atlanta, Ga.).

Gnedenko, B. V. Locally stable distributions. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **6**, 291–308 (1942). (Russian. English summary) [MF 8502]

Detailed proofs of the results announced previously [C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 263–266 (1942); these Rev. **4**, 102]. The following theorem has not been referred to before. The variable $\xi(t)$ follows the law of the iterated logarithm if, and only if, $t^{\frac{1}{2}}\xi(t)-b_t$ tends, for $t\to 0$, to a Gaussian distribution. *W. Feller.*

Gnedenko, B. Sur la croissance des processus stochastiques homogènes à accroissements indépendants. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **7**, 89–110 (1943). (Russian. French summary) [MF 9581]

Detailed proofs of the results announced previously [C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 3–4 (1942); these Rev. **4**, 103]. *W. Feller* (Providence, R. I.).

Fortet, Robert. Sur le calcul de certaines probabilités d'absorption. C. R. Acad. Sci. Paris **212**, 1118–1120 (1941). [MF 9186]

In a previous note [same C. R. **212**, 325–326 (1941); these Rev. **3**, 4] the author has studied various properties of the random variable $X(t)$ belonging to the parabolic differential equation (*) $u_t+u_{xx}+b(t,x)u_x=0$. Now let C be a continuous curve defined by $x=\phi(t)$. The author defines two "probabilities of absorption": $P_C(t,x;\tau)$ denotes the conditional probability that $X(t')<\phi(t')$ for all $t\leqq t'\leqq\tau$, if it is known that $X(t)=x$; and $\bar{P}_C(t,x;\tau)$ denotes the corresponding probability with the inequality replaced by $X(t')\leqq\phi(t')$. The author states that $\bar{P}_C=P_C$. He characterizes P_C in various ways, states some continuity properties and finds that P_C is a solution of (*). No proofs are given, but it is stated that S. Bernstein's theory of stochastic differential equations is used. The results are of interest not only for the theory of probability but also from the viewpoint of differential equations [cf. the author's note in the same C. R. **213**, 553–556 (1941); these Rev. **5**, 123].
W. Feller (Providence, R. I.).

Ville, Jean. Sur un problème de géométrie suggéré par l'étude du mouvement brownien. C. R. Acad. Sci. Paris **215**, 51–52 (1942). [MF 9477]

The author proves that, if $x(t)$, $y(t)$, $z(t)$ are the coordinates of a particle in a Brownian movement, under the usual hypotheses (x, y, z displacements in time t mutually independent and Gaussian, with means 0 and dispersions

$\sigma^2 h$), then, with probability 1, the particle will never pass through ($x(0)$, $y(0)$, $z(0)$) again, when t increases from 0. Although the proof is only valid in three dimensions, the author conjectures that the theorem is also true in two dimensions. *J. L. Doob* (Washington, D. C.).

Lévy, Paul. Intégrales stochastiques. C. R. Acad. Sci. Paris **212**, 1066–1068 (1941). [MF 9636]

Let $f(t)$, $g(t)$ be defined and continuous in $I: 0 \leqq t \leqq T$. Let $T_1, T_2, \cdots$ be mutually independent chance variables, each distributed uniformly over I. Let $t_1^{(n)}, \cdots, t_n^{(n)}$ be $T_1, \cdots, T_n$ arranged in order, with $t_0^{(n)}=0$, $t_n^{(n)}=T$. The author discusses conditions under which the sums

$$\sum_{j=1}^{n} [g(t_{j-1})+g(t_j)][f(t_j)-f(t_{j-1})]$$

converge with probability 1 to a limit, the "stochastic integral" $\int_0^T f(t)dg(t)$. It is shown that the superior and inferior limits are constant, with probability 1.

J. L. Doob (Washington, D. C.).

Mathematical Statistics

***"Student's" Collected Papers.** Edited by E. S. Pearson and John Wishart. Biometrica Office, University College, London, 1942. xiv+224 pp. 15 shillings.

The volume consists of the 21 papers William Sealy Gosset published under the pseudonym "Student" (14 of them in Biometrika), preceded by a brief biographical sketch by Launce McMullen, and followed by 3 letters to Nature and 4 contributions to discussions at meetings of the Royal Statistical Society. The articles deal with the distributions of sample means, sample variances and the Student ratio, applications of the Poisson distribution, the theory of field trials (6 papers), correlation theory, errors of routine analysis, evolution by selection.

Mathematically naïve, Student's writings nevertheless have already taken their place among the classics of mathematical statistics. The reasons lie not only in the historical importance of the pioneer papers on small sample theory, but also in Student's fine sense of the interrelation of theory and practice. Besides this there will grow on the reader a pleasant and unusual experience from a collection of scientific papers: an awareness of a likeable, modest and humorous human being. *H. Scheffé.*

Nair, K. R. and Banerjee, K. S. A note on fitting of straight lines if both variables are subject to error. Sankhyā **6**, 331 (1943). [MF 9082]

The authors consider the method of group averages for fitting a straight line when both variables x and y are subject to error. This method can be described as follows. Let $p_\alpha=(x_\alpha, y_\alpha)$ ($\alpha=1, \cdots, N$) be the observed pairs arranged in order of increasing magnitude of x. For fitting a straight line the first third and the last third of the points p_α ($\alpha=1, \cdots, N$) are plotted and the points of mean values of these two groups are joined by a straight line. This method is compared with the method of averages according to which the point of mean values of the first half of the observed pairs is joined with that of the second half. The numerical results of a model sampling carried out by the authors seem to indicate that the method of group averages gives a more accurate estimate of the "true line" than the method of averages. No theoretical discussion of why this should be so is given. *A. Wald* (New York, N. Y.).

Dugué, Daniel. Sur un nouveau type de courbe de fréquence. C. R. Acad. Sci. Paris **213**, 634–635 (1941). [MF 9644]

The new type, called "type harmonique," is given by

$$y=Ax^{-1}\exp\{-ax-b/x\},$$

where the constant A of integration is given by a Bessel function. It is stated that, from a sample, the parameters a and b can be found by arithmetic and geometric means; the statistics are exhaustive. The author states that he was led to the new curves from both theoretical and practical considerations; he has applied it with success.

W. Feller (Providence, R. I.).

Bhattacharyya, B. C. The use of McKay's Bessel function curves for graduating frequency distributions. Sankhyā **6**, 175–182 (1942). [MF 8637]

A. T. McKay [Biometrika **24**, 39–44 (1932)] suggested the use for graduation purposes of curves of the type

$$y=y_0\cdot e^{-cx/b}\cdot|x|^m\cdot\{\pi I_m\cdot|x/b| \quad\text{or}\quad K_m\cdot|x/b|\},$$

where I_m and K_m are Bessel functions defined by G. N. Watson [A Treatise on the Theory of Bessel Functions, Cambridge, 1922]. McKay has shown how to calculate the parameters for those curves, and he gave a comparison of their applicability with that of the curves of the Pearson system. In the present paper this comparison is carried to a greater detail, and a set of empirical data is actually fitted by a Pearson type IV curve and by a McKay curve. The paper is difficult to read due to a great number of misprints [on the first page six essential formulas are misprinted].

Z. W. Birnbaum (Seattle, Wash.).

Shrivastava, M. P. Bi-variate correlation surfaces. Science and Culture **6**, 615–616 (1941). [MF 9286]

The author sets down nine bivariate correlation functions whose marginal distributions belong to the Pearson system or are certain Bessel distribution laws. *C. C. Craig.*

Radhakrishna Rao, C. On bivariate correlation surfaces. Science and Culture **8**, 236–237 (1942). [MF 9341]

Roy, S. N. and Bose, Purnendu. The distribution of the root-mean-square of the second type of the multiple correlation co-efficient. Science and Culture **6**, 59 (1940). [MF 9283]

It is announced that the authors have found the distribution of the root-mean-square of the multiple correlation coefficient R in m samples of n from a p-dimensional normal universe, all of the variables being subject to error. The distribution function is set forth and it is seen to be of the same form as that for R itself. *C. C. Craig.*

Worcester, Jane and Wilson, Edwin B. A table determining L.D.50 or the fifty per cent end-point. Proc. Nat. Acad. Sci. U. S. A. **29**, 207–212 (1943). [MF 8714]

Hussain, Q. M. A note on interaction. Sankhyā **6**, 321–322 (1943). [MF 9080]

In a two-way table the expectation of the element in the jth row and ith column is given by $\mu+\alpha_i+\beta_j$. If a tetrad of 2×2 cells is defined to be the 4 cells determined by the row subscripts j and j' and the column subscripts i and i'

and if the tetrad difference is defined to be the element in the ith column and jth row plus the element in the i'th column and j'th row minus the element in the ith column and j'th row minus the element in the i'th column and jth row, then the expectation of the tetrad is equal to zero if there is no interaction between the row and column effects. It is shown that the interaction variance can be expressed in terms of these tetrads and that each individual tetrad may be tested for significance of difference from zero by t or F distributions. It is mentioned that by using the logarithms of the frequencies the same types of considerations apply to a $p\times q$ contingency table. *W. G. Madow.*

Craig, Allen T. Note on the independence of certain quadratic forms. Ann. Math. Statistics **14**, 195–197 (1943). [MF 8780]

Several contributions are made to the study of quadratic forms in normally and independently distributed variates. The first of these states that a necessary and sufficient condition for the statistical independence of two real symmetric quadratic forms in n normally and independently distributed variates is that the product of the matrices be zero. The second states that a necessary and sufficient condition for two real symmetric quadratic forms in n normally and independently distributed variates to be independently distributed in chi-square distributions is that the product of the matrices be zero, and that each matrix equal its own square. These theorems are readily extended to the case of k quadratic forms. Finally, it is shown that, if the sum of the matrices of k quadratic forms is equal to the identity matrix, then the condition for the independence of the quadratic forms already derived is necessary and sufficient for the quadratic forms to be independently distributed in chi-square distribution. This results from the fact that the equality of the matrices and their squares is a consequence of the vanishing of the product of the matrices and the condition that the sum of the matrices be the identity matrix. *W. G. Madow* (Washington, D. C.).

Guttman, Louis and Cohen, Jozef. Multiple rectilinear prediction and the resolution into components. II. Psychometrika **8**, 169–183 (1943). [MF 9065]

This paper is a continuation of a previous paper by Guttman [Psychometrika **5**, 75–99 (1940); these Rev. **2**, 234], being devoted to numerical methods based on the theory developed in the earlier paper. Advantage is taken of the circumstance that the number of common factors may be considerably less than the number of tests, the calculations being carried out directly from the factor loadings. Systematic procedures are laid out for determining the regressions of one test upon the remaining ones, of a common factor upon the tests and of a unique factor upon the tests. A work sheet is given with explicit directions for each step of the calculations and the whole is illustrated with a numerical example. The process stems from a computation of the universe of the correlation matrix and it should be compared with Dwyer's use of his abbreviated Doolittle method. [Cf. Psychometrika **5**, 211–232 (1940); **6**, 355–365 (1941); these Rev. **2**, 234; **3**, 154.] *C. C. Craig.*

Greenwood, J. A. A preferential matching problem. Psychometrika **8**, 185–191 (1943). [MF 9066]

The author discusses a statistic which might be described as subjective correlation and which admits of fairly wide applicability. Thus there exists no objective measure of the correlation between handwriting of identical twins but it is possible to measure the ability of a judge to detect the similarity. Handwriting specimens are written on cards and the cards are separated into two decks so that the two specimens from a pair of twins never appear in the same deck. The judge assigns to each pair of cards taken one from each deck an integer expressing the degree of similarity. This is called preferential matching. The matching score is the trace of the matrix thus constructed, that is, the sum of the integers assigned to those pairs which are handwriting specimens of twins. The desired statistic is the probability that chance would produce a score as good as that which the judge attained. This device was first suggested by C. E. Stuart. *A. H. Copeland.*

Smirnoff, N. On the estimation of the maximum term in a series of observations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 346–350 (1941). [MF 9568]

Let $X_1, \cdots, X_n$ be n independent observations of normally distributed random variables in ascending order of magnitude. Let $\bar{X}$ be the sample mean and S the sample variance. Then $\xi_n=(X_n-\bar{X})/S$ is the maximum normed deviation in the sample. The author finds the cumulative distribution function $\tau_n(\lambda)$ of ξ_n. For any fiducial probability α the equation $\tau_n(\lambda)=\alpha$ determines the limits $\lambda_n(\alpha)$ of significance for the maximum normed deviation from the mean. Tables are given for $\lambda_n(\alpha)$ for $3\leqq n\leqq 20$ and $\alpha=.9$, .925, .950, .975. It is stated that, for small n, the limits $\lambda_n(\alpha)$ do not differ much from those obtained by W. R. Thompson [Ann. Math. Statistics **6**, 214–219 (1935)]. *W. Feller* (Providence, R. I.).

Lawley, D. N. A note on Karl Pearson's selection formulae. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 28–30 (1943). [MF 9666]

Given an n-variate normally distributed population, Pearson's selection problem was concerned with the distribution law for that subsection of the population in which p selected variables obeyed a given normal distribution law, specified, of course, by p means and a pth order variance matrix. Pearson's solution was later obtained more elegantly by Soper and particularly by Aitken [Proc. Edinburgh Math. Soc. (2) **4**, 106–110 (1935)]. In the present paper the author points out, using the vector notation introduced by Aitken, that the means vector and variance matrix previously found in the case of a normal universe are still valid if the regressions of the unselected variables on the selected ones are linear and the form of the distribution of the unselected variables for given values of the selected ones is independent of those values except for the means. It is further shown that these results hold if and only if the first and second degree terms in the expansion of the moment generating function of the unselected variables are independent of the values of those selected. *C. C. Craig.*

Villars, D. S. and Anderson, T. W. Some significance tests for normal bivariate distributions. Ann. Math. Statistics **14**, 141–148 (1943). [MF 8775]

The study makes use of the fact that, when sufficiently restrictive assumptions are made, the study of a normal multivariate distribution becomes essentially the study of a univariate normal distribution. The assumptions generally made are that the variances are equal or their ratios are known, and that the correlations are zero or their values

are known. Various results are obtained by use of such assumptions. In addition, relation to principal components of the fitting of a straight line in such a way that the sum of squares of distances from the sample points to the line is a minimum is again demonstrated. *W. G. Madow.*

Simon, Herbert A. Symmetric tests of the hypothesis that the mean of one normal population exceeds that of another. Ann. Math. Statistics **14**, 149–154 (1943). [MF 9535]

Let $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ be normally and independently distributed with the same variances but with the mean value of x_i being M_x and the mean value of y_i being M_y $(i=1, \cdots, n)$. A region E of the $2n$ dimensional sample space is called a symmetric test of the hypothesis $H_0=M_y \geqq M_x$ if, whenever the observation $(a_1, \cdots, a_n; b_1, \cdots, b_n)$ is an element of E, the observation $(b_1, \cdots, b_n; a_1, \cdots, a_n)$ is an element of the complement of E, where the first n values of (   ) are values of $x_1, \cdots, x_n$ and the second n value of (   ) are values of $y_1, \cdots, y_n$. If a symmetric test T_s of a hypothesis H_0 is more powerful than any other symmetric test of H_0 then T_s is the uniformly most powerful symmetric test of H_0. It is shown that the region $\bar{y} > \bar{x}$ is the uniformly most powerful test of the hypotheses H_0. When the variances are unequal, it is shown that no uniformly most powerful symmetric test of H_0 exists. However, a uniformly most powerful test does exist if x_i and y_i can be paired. Then observations become $z_i = y_i - x_i$, and H_0 becomes "the mean value of z_i is greater than or equal to zero, $i=1, \cdots, n$." The uniformly most powerful symmetric test is $\bar{z} > 0$. *W. G. Madow.*

Mathisen, Harold C. A method of testing the hypothesis that two samples are from the same population. Ann. Math. Statistics **14**, 188–194 (1943). [MF 8779]

A method of testing the hypothesis that two samples are from the same population is given. This method may be described as follows. Denote by m_1 the number of observations in the second sample which have values lower than the median of the first sample. If the population distribution is continuous, the distribution of m_1 does not depend on the latter. Thus lower and upper critical values of m_1 can be given for any level of significance. The author gives a table of these critical values for different sample sizes. The method is extended to the case of four intervals determined by the medium and the two quartiles of the first sample. If the second sample is of size $4m$ and if m_i denotes the number of observations in the second sample which fall in the ith interval $(i=1, 2, 3, 4)$, the proposed test function is given by

$$C = \left[\sum_{i=1}^{4} (m_i - m)^2\right] / gm^2.$$

A Pearson type I curve is fitted to the distribution of C by equating the first two moments and significance points are computed. *A. Wald* (New York, N. Y.).

Festinger, Leon. A statistical test for means of samples from skew populations. Psychometrika **8**, 205–210 (1943). [MF 9680]

The skew populations considered are those known to have a Pearson type III probability density function $f(x) = cx^{p-1}e^{-bx}$, $0 < x < +\infty$, with known p and unknown b. Well-known sampling theory for the sample mean then yields tests and confidence intervals for the population mean or for the difference of two population means. *H. Scheffé.*

Sankara Pillai, K. A note on Poisson distribution. Proc. Indian Acad. Sci., Sect. A. **18**, 179–189 (1943). [MF 9714]

This paper considers the problem of estimating the mean m of a Poisson distribution, given y_k $(k=0, 1, \cdots, n)$, the number of observations in which the chance variable X took the value k. If Y_k is the probability that $X=k$, we obviously have $Y_{k+1} = (m/(k+1))Y_k$. The author proposes the estimate m_2 obtained by minimizing

$$\sum_{k=0}^{n-1} (y_{k+1} - (m/(k+1))y_k)^2,$$

a procedure which he considers a "direct application of the least square principle." He then obtains

$$m_2 = \left(\sum_{k=0}^{n-1} \frac{y_k y_{k+1}}{k+1}\right) \Big/ \left(\sum_{k=0}^{n-1} \frac{y_k^2}{(k+1)^2}\right)$$

and its asymptotic variance. He shows that m_2 is unbiased ($E(m_2) = m$; the author calls it consistent), and consistent (m_2 converges stochastically to m with increasing sample size N; this follows from the fact that the asymptotic variance is of order $1/N$), and that its asymptotic variance is substantially larger than that of the maximum likelihood estimate m_1.

Since m_2 is asymptotically less efficient than m_1 and is not simpler to compute, and since the author has not demonstrated any small sample property in which m_2 is superior to m_1, it is the opinion of this reviewer that a cogent argument for the use of m_2 as opposed to that of m_1 has not yet been adduced. *J. Wolfowitz.*

Curtiss, J. H. On transformations used in the analysis of variance. Ann. Math. Statistics **14**, 107–122 (1943). [MF 8772]

Many tests of significance in current use, especially those in connection with the analysis of variance and regression problems, are based on the a priori assumption that the observed variates are independently and normally distributed with a common variance. However, sometimes we have to deal with nonnormal distributions in which the variance is functionally related to the mean. To make the normal theory applicable in such cases, it is necessary to replace each variate X by a transform $f(X)$ so that normality of the distribution and stability of the variance are achieved as nearly as possible. In this paper a theory of a class of such transformations is given and applied to a number of transformations in current use. Three asymptotic theorems are derived, the first of which states that under certain restrictive conditions the limiting distribution of $Y = (X - \mu_n)\psi_n(\mu_n)$ is the same as the limiting distribution of $W = f(X) - f(\mu_n)$, where μ_n is the mean of X, $\psi_n(x) \geqq 0$, $f(X) = \int_a^X \psi_n(x)dx$ and a is an arbitrary constant. In the subsequent applications of the theory to the square root transformation for a variate with a Poisson distribution and for a variate with a Γ distribution and in the application to the inverse sine transformation for a binomial variate, the function $\psi_n(\mu_n)$ is proportional to the reciprocal of the standard deviation of X and the limiting distribution of Y is normal. Also the logarithmic transformation is discussed; it differs in certain respects from the transformations previously considered. *A. Wald* (New York, N. Y.).

Wald, Abraham. On the power function of the analysis of variance test. Ann. Math. Statistics **13**, 434–439 (1942). [MF 7874]

Let $x_1, \cdots, x_n$ be mutually independent normal variables with common (unknown) variance σ^2 and the expectations

$$E(x_i)=\sum_{j=1}^{s} a_{ij}\vartheta_j,$$

where the matrix $\|a_{ij}\|$ is nonsingular. Furthermore, let H denote the hypothesis that each of the $r\leqq s$ independent functions

$$\Theta_i=\sum_{j=1}^{s} b_{ij}\vartheta_j, \qquad t=1, 2, \cdots, r,$$

has a specified value Θ_i^0. It is known [P. C. Tang, Statist. Research Mem. London **2**, 126–149 (1938)] that the power function of the analysis of variance test of this hypothesis depends on the unique argument $\lambda=(1/\sigma^2)Q$, where Q is a quadratic form in the difference $\eta_j=\Theta_i-\Theta_i^0$. P. L. Hsu has proved [Biometrika **32**, 62–69 (1941); these Rev. **2**, 236] that, out of all the tests corresponding to the same level of significance, the power function of which depends only on the argument λ, the analysis of variance test is the most powerful one.

The paper under review gives a result generalizing the above theorem of Hsu. Consider the space of $\Theta_1, \Theta_2, \cdots, \Theta_r$ and denote by S the hypersurface in this space determined by $Q=$const. Consider any test T of the hypothesis H based on some critical region w. Let further $\gamma_w(S)$ denote the average value of the power function of the test T over the hypersurface S. It is then proved that, if the test T and the analysis of variance test correspond to the same level of significance, then the power function of the analysis of variance test is at least equal to $\gamma_w(S)$ on almost all S. The proof is based on two lemmas. The second of these lemmas, meant to prove a structural property of the regions similar to the sample space with respect to the parameters unspecified by H, is unnecessary because this property is a direct consequence of the known general results on this subject. Also its published proof does not seem to apply if the lemma is taken as generally as it is stated.

J. Neyman (Berkeley, Calif.).

Wald, Abraham. On the efficient design of statistical investigations. Ann. Math. Statistics **14**, 134–140 (1943). [MF 8774]

Let $y_1, y_2, \cdots, y_N$ be N independent normal variates having the same variance σ^2. The expectations of the y's are supposed to be linear combinations of p parameters $\beta_1, \beta_2, \cdots, \beta_p$, the values of which are unknown:

$$E(y_\alpha)=\beta_1x_{1_\alpha}+\beta_2x_{2_\alpha}+\cdots+\beta_px_{p_\alpha}$$

where the coefficients x_{i_j} are calculable from the arrangement of the experiment designed to yield the particular values of the variates y_α. Following Kołodziejczyk [Biometrika **27**, 161–190 (1935)], the author uses the term "linear hypothesis" to describe the hypothesis H that $r\leqq p$ linear combinations of the parameters $\Theta_i=g_{i1}\beta_1+\cdots+g_{ip}\beta_p$, with known coefficients g_{ij}, have some specified values Θ_i^0, for $i=1, 2, \cdots, r$. As was shown by Tang [Statist. Res. Mem. London **2**, 126–149 (1938)], the power of the likelihood ratio test of the most general linear hypothesis is an increasing function of a single parameter $\lambda=Q/\sigma^2$, where Q denotes a quadratic form in the differences $\nabla_i-\Theta_i^0$. The coefficients of the form Q are calculable from the values of the x_{i_j} and hence, ultimately, from the design of the experiment which could yield the values of the y's. The author considers possible definitions of the measure of efficiency of an experiment, designed to test a linear hypothesis. The first and the most natural idea is to classify the efficiency of the experiments according to the power of the likelihood ratio test of H and thus, ultimately, according to the value of the quadratic form Q. If $r=1$, then there exists a single system of coefficients x_{i_j} and, therefore, a calculable design of the experiment which maximizes Q, irrespective of the values of the differences $\Theta_i-\Theta_i^0$. On the other hand, if $r>1$, the design of the experiment which maximizes Q with one system of values of the differences $\Theta_i-\Theta_i^0$ would not necessarily do so with another. The author's second thought is to measure the efficiency of the experimental design by the minimum value of Q attained on the sphere $\sum(\Theta_i-\Theta_i^0)^2=1$. In view of some difficulties, the final definition of the efficiency is different, to the reviewer's mind somewhat arbitrary and perhaps too long to explain in a review. It appears that in the sense of this definition the Latin square and the Graeco-Latin square designs are more efficient than any other experimental designs of certain classes.

J. Neyman (Berkeley, Calif.).

Mann, H. B. and Wald, A. On the statistical treatment of linear stochastic difference equations. Econometrica **11**, 173–220 (1943). [MF 9250]

For any integral value of t let $x_{1t}, \cdots, x_{rt}$ be a set of r random variables which satisfy the system of linear stochastic difference equations

$$\sum_{j=1}^{r}\sum_{k=0}^{p_{ij}} \alpha_{ijk}x_{j,t-k}+\alpha_i=\epsilon_{it}, \qquad i=1, \cdots, r.$$

The coefficients α_{ijk}, α_i are known or unknown constants and the $\epsilon_t=(\epsilon_{1t}, \cdots, \epsilon_{rt})$ are independently distributed random vectors each having the same distribution. It is assumed that $E(\epsilon_{it})=0$ and that the determinant of the covariance matrix of $\epsilon_{1t}, \cdots, \epsilon_{rt}$ is not zero. It is also assumed that all moments of ϵ_{it} are finite and that the roots of the determinantal equation $|b_{ij}(\rho)|=0$ in ρ, where $b_{ij}(\rho)=\sum_{k=0}^{p_{ij}}\alpha_{ijk}\rho^{-k}$, are less than 1 in absolute value. The authors deal with the problem of setting up statistical estimates for the unknown α's and for some of the unknown parameters of the distribution of the random vector ϵ_t, on the basis of the values of x_{it}, observed for a finite number of "time" points t. The estimates proposed by the authors are equal to the maximum likelihood estimates if the vector ϵ_t is normally distributed. The joint limiting distribution of these estimates when the number N of observations approaches infinity is derived without assuming normality of the distribution of ϵ_t. From the joint limiting distribution of the estimates, which is shown to be multivariate normal, confidence limits are derived for the unknown α's and the unknown covariances σ_{ij} of ϵ_{it} and ϵ_{jt}. If the joint distribution of $\epsilon_{1t}, \cdots, \epsilon_{rt}$ is normal the limiting distribution of the maximum likelihood estimate can be obtained as follows. Let m be the number of unknown parameters, that is, the number of unknown coefficients α_{ijk}, α_i and unknown covariances σ_{ij}. Arrange these parameters in an ordered sequence and denote them by $\theta_1, \cdots, \theta_m$, respectively. Let $\bar\theta_1, \cdots, \bar\theta_m$ be the maximum likelihood estimates of $\theta_1, \cdots, \theta_m$, respectively. Denote by p the joint probability density function of the observations x_{it} $(i=1, \cdots, r;$

$t=1,\cdots,N$) and let $\bar{p}$ be the expression we obtain from p if we replace θ_i by $\bar{\theta}_i$ ($i=1,\cdots,m$). Then, if certain conditions for the consistency of the maximum likelihood estimates are satisfied, the distribution of the variates $\xi_u=\sqrt{N}(\bar{\theta}_u-\theta_u)$ ($u=1,\cdots,m$) approaches, with $N\to\infty$, a multivariate normal distribution with zero means. The limit of the covariance matrix $\|\sigma(\xi_u\xi_v)\|$ ($u,v=1,\cdots,m$) is equal to the stochastic limit of

$$\left\|-\frac{1}{N}\frac{\partial^2\log\bar{p}}{\partial\bar{\theta}_u\partial\bar{\theta}_v}\right\|^{-1}.$$

The quantity $(\partial^2\log\bar{p})/(\partial\bar{\theta}_u\partial\bar{\theta}_v)$ can be calculated from the observations. *J. Wolfowitz* (New York, N. Y.).

Dodge, H. F. A sampling inspection plan for continuous production. Ann. Math. Statistics **14**, 264–279 (1943). [MF 9144]

A plan for sampling inspection of units of a product, manufactured in quantity by an essentially continuous process, is described. The inspection is assumed nondestructive, free of error and such as to sort the unit inspected into two classes, "defective" and "nondefective." It is also assumed that the manufacturing process is "statistically controlled so that the probability (p) of producing a defective unit is constant." The plan consists of the following steps: (a) inspection of all units until i consecutive ones are found nondefective; (b) thereafter inspection of a fraction f of the units until a defective one is found; (c) repetition of the process described in (a) and (b). (Defective units are of course removed.) The maximum average fraction (over a long series) of defectives in a product thus inspected (called AOQL, average outgoing quality limit) is obtained as a function of i and f. Graphs of the AOQL for various combinations of i and f are given, and the effects of the variation in i and f discussed.

The remainder of the paper deals with practical aspects of sampling inspection, and with the use of the plan when several characteristics of the product are to be inspected. *J. Wolfowitz* (New York, N. Y.).

Nair, K. Raghavan and Radhakrishna Rao, C. A general class of quasi-factorial designs leading to confounded designs for factorial experiments. Science and Culture **7**, 457–458 (1942). [MF 9547]

Mahalanobis, P. C. Mathematics and statistics. Sample surveys. Science and Culture **7**, Suppl. 1–2 (1942). [MF 9548]

Sinha, H. Role of mathematics in economic statistics. Science and Culture **6**, 255–258 (1940). [MF 9672]

Wold, Herman. A synthesis of pure demand analysis. Skand. Aktuarietidskr. **1943**, 85–118 (1943). [MF 8850]

Three lines of approach can be distinguished in the modern theories of demand analysis which do not postulate the measurability of utility. These are (1) the "demand function" approach which starts with the explicit assumption that the demanded quantities are certain single valued functions of income and prices, (2) the approach based on the notion of indifference surfaces, (3) the so-called "marginal substitution" approach. One of the main objects of the present paper is to show that (1) certain integrability conditions must be included among the postulates of the "demand function" and the "marginal substitution" approaches, since otherwise the resulting demand structure would not be consistent; (2) the three approaches become equivalent if the integrability conditions are added to the postulates of the "demand function" and "marginal substitution" theories. These statements are verified in part I of the paper, except for those referring to the demand function approach. The discussion of the latter is given in part II which is not available to the reviewer. *A. Wald.*

RELATIVITY

Lichnerowicz, André. Sur l'intégration des équations d'Einstein. C. R. Acad. Sci. Paris **213**, 516–518 (1941). [MF 9171]

This paper deals with the question: what boundary conditions define a solution of Einstein's field equations in empty space-time? The boundary conditions are to be assigned on a hypersurface S with equation $x^4=0$. This S is chosen to be a minimal surface (sum of principal curvatures equal to 0). Without loss of generality, we can write (throughout space-time) $ds^2=(Vdx^4)^2+g_{ij}dx^idx^j$, so that the unknowns are V and g_{ij}. They are subject to the field equations $S_{\alpha\beta}=0$, which split into two sets $S_{ij}=0$ and $S_\alpha^4=0$. (Greek suffixes are 1, 2, 3, 4; Latin suffixes are 1, 2, 3.) The equations $S_{ij}=0$ merely control the variation of the unknowns with x^4. The essential consistency conditions are $S_\alpha^4=0$; these equations are satisfied everywhere if they are satisfied on S. The minimal condition on S simplifies the equations.

An arbitrary tensor g^*_{ij} is assigned on S, and a scalar θ and a tensor Π_{ij} are introduced. The values of g_{ij} on S and its normal derivative $\partial_4 g_{ij}$ are then written $g_{ij}=e^{2\theta}g^*_{ij}$, $\partial_4 g_{ij}=2Ve^{-2\theta}\Pi_{ij}$. The conditions $S_\alpha^4=0$ subject θ and Π_{ij} to certain partial differential equations. The author discusses the extent to which Π_{ij} may be assigned on a 2-space bounding a region of S and points out that the equation for θ is of elliptic type and admits a uniqueness theorem. [For a different approach to the same general problem, see Lanczos, Phys. Z. **23**, 537–539 (1922) and A. Finzi, Atti Accad. Naz. Lincei. Rend. (6) **27**, 324–330 (1938).] *J. L. Synge* (Columbus, Ohio).

Lichnerowicz, André. Sur l'intégration des équations de la relativité. C. R. Acad. Sci. Paris **213**, 549–551 (1941). [MF 9173]

The method of the preceding paper is applied to the determination of the motion of a homogeneous incompressible fluid in general relativity. *J. L. Synge.*

Lichnerowicz, André. Opérateurs hermitiques et espace de Riemann. C. R. Acad. Sci. Paris **213**, 12–14 (1941). [MF 9150]

This paper is an attempt to reconcile conditions of quantization with invariance under coordinate transformations in Riemannian space, with a view to quantization of the motion of a system or of a point in a relativistic universe. Here X^α and Y_α are two sets of Hermitian matrices, each set commuting in itself; $G^{\lambda\mu}$ are Hermitian matrices which are functions of X^α and so commute with one another and with X^α. A Hamiltonian $H=\frac{1}{2}Y_\lambda G^{\lambda\mu}Y_\mu$ yields canonical equations $dX^\alpha/du=\partial H/\partial Y_\alpha$,

$dY_\alpha/du = -\partial H/\partial X^\alpha$. In a "coordinate transformation" $X^\alpha = f^\alpha(X^{\alpha'})$, we have $G^{\lambda\mu} = A_{\lambda'}{}^{\lambda} A_{\mu'}{}^{\mu} G^{\lambda'\mu'}$, $Y_\alpha = A_\alpha{}^{\alpha'} Y_{\alpha'}$, where $A_\alpha{}^{\alpha'} = \partial X^{\alpha'}/\partial X^\alpha$. The commutation relations for the pairs $(G^{\lambda\mu}, Y_\alpha)$, $(dX^\alpha/du, Y_\beta)$, $(dX^\alpha/du, dX^\beta/du)$ are subjected to two conditions: (a) they must be invariant to terms of order h^2 at least; (b) if in a system of reference $\partial G^{\lambda\mu}/\partial X_\alpha$ are all zero at a point, at that point the operators commute. The author claims that the following commutation conditions follow:

$$G^{\lambda\alpha} Y_\alpha - Y_\alpha G^{\lambda\alpha} = -2\bar{h} H^\lambda,$$

$$\frac{dX^\alpha}{du} Y_\beta - Y_\beta \frac{dX^\alpha}{du} = \bar{h} \Gamma_{\beta\rho}^{\alpha} \frac{dX^\rho}{du},$$

$$\frac{dX^\alpha}{du}\frac{dX^\beta}{du} - \frac{dX^\beta}{du}\frac{dX^\alpha}{du} = 0,$$

where $H^\lambda = \frac{1}{2}\partial G^{\alpha\lambda}/\partial X^\alpha$, $\Gamma_{\beta\rho}{}^\alpha = G^{\alpha\sigma}[{}^{\beta}{}_{\sigma}{}^{\rho}]$ and $\bar{h}$ is a constant of the order of h. When these conditions are used in the canonical equations, they become formally similar to the usual geodesic equations. For the case of an electromagnetic field, the author refers to an earlier paper [same C. R. 212, 328–331 (1941); these Rev. 3, 63]. *J. L. Synge.*

Pendse, C. G. On null geodesics and null-corpuscles in the theory of relativity. Philos. Mag. (7) **34**, 377–393 (1943). [MF 8734]

The author proposes to replace the usual definition of a null geodesic (that is, the limiting curves of the family satisfying $\delta\int d\bar{s}=0$ such that $ds=0$ along them) by the following: the curves such that $\delta\int(ds/dt)^2 dt=0$ and $ds/dt=0$. He argues that the extraneous parameter t which is introduced is essential and suggests that it is connected with energy. *A. H. Taub* (Princeton, N. J.).

Roubaud-Valette, Jean. Le groupe de Lorentz et les espaces généralisés. C. R. Acad. Sci. Paris **212**, 1131–1134 (1941). [MF 9190]

By considering the generalized Lorentz group extended to include rotations of space, the author seeks to arrive at the possible forms of space for the five-dimensional relativity theory he has proposed elsewhere [C. R. Acad. Sci. Paris **208**, 1556–1558 (1938)]. For example, he concludes that the three-dimensional subspace determined by the coordinates t, u and $r=(x^2+y^2+z^2)^{\frac{1}{2}}$, where u is the fifth dimension, is elliptic. *O. Frink* (State College, Pa.).

Roubaud-Valette, Jean. Sur l'édification d'une géométrie ondulatoire. C. R. Acad. Sci. Paris **214**, 791–794 (1942). [MF 9453]

Roubaud-Valette, Jean. Sur l'édification d'une géométrie ondulatoire. C. R. Acad. Sci. Paris **215**, 173–175 (1942). [MF 9485]

These are two notes on the construction of a "wave geometry" based on the author's five-dimensional relativity theory. The first note contains some rather disconnected remarks concerning the author's machinery of tensors, vectors, semi-vectors, dyads, dyadics and operators. The second note is concerned with the transition from the ordinary differential equations of motion of classical type for a charged particle in an electromagnetic and gravitational field, as derived in a previous paper, to wave-mechanical partial differential equations of Dirac or Gordon type. *O. Frink* (State College, Pa.).

Gutierrez Novoa, L. On the graphical representation of the geometry of space-time in the restricted theory of relativity. Bol. Mat. **16**, 139–144 (1943). (Spanish) [MF 9918]

Dingle, Herbert. The time concept in restricted relativity. Amer. J. Phys. **10**, 203–205 (1942). [MF 9279]

Epstein, Paul S. The time concept in restricted relativity—a rejoinder. Amer. J. Phys. **10**, 205–208 (1942). [MF 9295]

Infeld, L. Clocks, rigid rods and relativity theory. Amer. J. Phys. **11**, 219–222 (1943). [MF 9280]

Dingle, Herbert. The time concept in restricted relativity. Amer. J. Phys. **11**, 228–230 (1943). [MF 9281]

A discussion connected with Epstein's article in the same J. **10**, 1–6 (1942); these Rev. **3**, 291.

Dive, Pierre. Sur le groupe de déplacement euclidien dans la théorie de la relativité et les critériums expérimentaux. C. R. Acad. Sci. Paris **215**, 185–187 (1942). [MF 9486]

The author considers a Galilean coordinate system K and a moving coordinate system K^1. When the motion of K^1 is helical with respect to K, it is stated that results obtained by special relativity will agree with those obtained by classical mechanics. From this it is concluded that experiments involving motion of this kind cannot serve as criteria for the validity of special relativity. *M. Wyman.*

Milne, E. A. Note on the interaction of two point-charges. Philos. Mag. (7) **34**, 712–716 (1943). [MF 9559]

It is shown that the derivation of the integrals of energy and angular momentum for a pair of point charges can be done in a more elementary way than was done before by the author [Philos. Mag. (7) **34**, 73–101, 197–211, 235–258 (1943); these Rev. **4**, 226, 227, 285]. Indeed the same conclusions as before are now drawn from the relativistic equations of motion:

$$\frac{d}{d\tau}\frac{m\dot{\mathbf{r}}}{(1-v^2/c^2)^{\frac{1}{2}}} = \frac{e_1e_2}{(1-v^2/c^2)^{\frac{1}{2}}}\frac{\mathbf{r}+\frac{1}{2}(\mathbf{r}\times\dot{\mathbf{r}})\times\dot{\mathbf{r}}/c^2}{|\mathbf{r}|^3}.$$

The energy integral follows by multiplying these equations scalarly by $\dot{\mathbf{r}}$ and the momentum integral by multiplying them vectorially by $\mathbf{r}$. *L. Infeld* (Toronto, Ont.).

de Beauregard, Olivier Costa. Sur la théorie des moments cinétiques propres en relativité restreinte. J. Math. Pures Appl. (9) **21**, 267–275 (1942). [MF 9304]

The author outlines some of the properties of spin in quantum mechanics and desires to set up a postulational treatment for special relativity that will give these same properties. He sets up the analogue of spin density and shows that this must be represented by a tensor of the first or third order. Three possible postulates are considered; the first two lead to the nul results of classical physics, whereas the third gives results that are in agreement with Dirac's theory. *M. Wyman* (Edmonton, Alta.).

de Beauregard, Olivier Costa. Sur la dynamique des milieux doués d'une densité de moment cinétique propre. C. R. Acad. Sci. Paris **214**, 904–906 (1942). [MF 9464]

The author gives an axiomatic treatment of the dynamics of mediums possessing a spin angular momentum. In relativity the use of these axioms compels the use of a non-Riemannian geometry. *M. Wyman* (Edmonton, Alta.).

Reichenbächer, Ernst. Das kosmische Gravitationsgesetz. Z. Astrophys. **22**, 111–116 (1943). [MF 9905]

Markow, M. Das Mehrkörperproblem in der klassischen relativistischen Theorie. Acad. Sci. USSR. J. Phys. **7**, 42–47 (1943). [MF 8887]

The author considers the problem of "n" charged particles in an electromagnetic field. He desires to show that Dirac's idea, of assigning a distinct time "t_s" to each of the particles, can be carried over into classical relativity. Moreover, it is shown that a classical meaning can be given to Fermi's supplementary condition. There are unfortunately several misprints in the paper and the notation used is not always good, nor clear. The formula given for $\partial^2\varphi/\partial t^2$ on page 44 is wrong. However, this error is corrected when $\partial^2\varphi/\partial t^2$ is used. *M. Wyman* (Edmonton, Alta.).

v. Laue, M. Ein relativistischer Beweis für das Wiensche Verschiebungsgesetz. Ann. Physik (5) **43**, 220–222 (1943). [MF 9926]

MECHANICS

Beer, F. P. A plane representation of vectors and tensors. Amer. Math. Monthly **50**, 605–610 (1943). [MF 9851]

The author recalls two types of plane images of free vectors in space which have been used in graphical statics to treat systems of forces in space by plane constructions. He slightly modifies these images and defines a plane image for the ellipsoid (tensor) of inertia of a rigid body for axes through a fixed point. In terms of these images he shows how to find the moment of inertia of the body about any one of these axes. His construction requires just one ellipse and some straight lines. *P. Franklin.*

Kasner, Edward and DeCicco, John. Generalized dynamical trajectories in space. Duke Math. J. **10**, 733–742 (1943). [MF 9756]

A study of the differential geometry of fields of force in space which depend on the direction as well as the position. The differential equations of the trajectories as well as some of their geometric properties are obtained, also the condition that the trajectories belong to a field independent of position. Certain related curves, velocity systems, as well as the extension of the results for force fields depending on higher differential elements, are given. *P. Franklin.*

Jouguet, Émile. Remarques sur les vitesses critiques et la stabilité séculaire des systèmes à variable cachée. C. R. Acad. Sci. Paris **214**, 929–931 (1942). [MF 9465]

Completion of an earlier note [C. R. Acad. Sci. Paris **207**, 649–652 (1938)] by the same author. The notes deal with a dynamical system having one ignorable angular coordinate. In particular, they deal with small departures from a given motion in which the ignorable coordinate is a linear function of the time and the remaining coordinates are constants. The author discusses the effects of different choices of the parameters defining the varied motion upon the classification of the critical angular velocities, and upon the type of stability of the given motion. *L. A. MacColl.*

Coenen, P. A. Über das immer ruhende Drehpendel im ungleichförmig rotierenden Raum. Physica **9**, 50–52 (1942). [MF 9809]

Fokker, A. D. The rising top, experimental evidence and theory. Physica **8**, 591–596 (1941). [MF 9541]

The author finds experimental evidence that during the nearly steady motion of a top there is no sliding. This causes him to reject the explanation of rising by assuming friction at the peg. Instead he returns to the theory as given in Routh's "Advanced dynamics of rigid bodies," [5 ed., Macmillan, London, 1892,] for small oscillations. By considering the second order terms he deduces a rising effect. *P. Franklin* (Cambridge, Mass.).

Roy, Louis. Sur le frottement de roulement. C. R. Acad. Sci. Paris **213**, 601–604 (1941). [MF 9641]

Painlevé showed by examples that in some cases the use of the ordinary Coulomb law of sliding friction in dynamical problems makes the solutions of the problems either impossible or indeterminate [Leçons sur le frottement, Hermann & Fils, Paris, 1895]. The author of the present note discusses a simple problem involving rolling friction, in which the same kind of difficulties appear. It is pointed out that the difficulties can be overcome, in a way that has been suggested already for the case of sliding friction, by taking more realistic account of the physical facts. *L. A. MacColl* (New York, N. Y.).

Pugachev, V. S. The general problem of exterior ballistics for aviation bombs. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 41–48 (1943). (Russian. English summary) [MF 9724]

The author expresses the differential equations for the motion of the center of gravity of a falling bomb with y (the distance fallen) as the independent variable. The equations then do not contain t and x explicitly. He assumes that the horizontal and vertical components u and w of the velocity may be expressed as power series in c, where c is the ballistic coefficient ($c=0$ corresponds to the case of no resistance), and obtains approximate formulas neglecting higher powers of c than the first. For additional simplicity the resistance may be assumed proportional to the square of the velocity and the air density a linear function of the altitude. Analytical formulas are then obtained for the time of flight and for the horizontal carry. *W. E. Milne.*

Malovichko, A. K. Determination of bedding of a disturbing mass from horizontal gradients of gravity. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 399–401 (1941). [MF 9552]

Hydrodynamics, Aerodynamics, Acoustics

Kravtchenko, Julien. Sur le problème de représentation conforme de Helmholtz: cas d'un contour sans tangente. C. R. Acad. Sci. Paris **214**, 870–872 (1942). [MF 9461]

Continuing his investigation on the wake problem [J. Math. Pures Appl. (9) **20**, 35–234, 235–303 (1941); these Rev. **3**, 219; **4**, 58; C. R. Acad. Sci. Paris **214**, 464–466 (1942); these Rev. **4**, 175], the author now treats the two dimensional wake problem of incompressible nonviscous flow connected with the contour BC defined in the Oxy plane by $x=x(y)$, $y_1\leqq y\leqq y_2$, where y_1 and y_2 are the ordinates of the end points B and C of BC, supposed to be

bounded. The function $x(y)$ is subjected to a Lipschitz condition

$$|x(y)-x(y')|\leqq\cot\varphi\cdot|y-y'|,\quad 0<\epsilon\leqq\varphi\leqq\pi/2.$$

Thus BC has a finite length L. The arc BC can be approached by a sequence of inscribed polygons P_n of n segments. The length of each segment decreases as $1/n$; thus $\lim P_n=BC$. If P_n has an inclination $\psi_n(l)$, l being the length along the contour, which satisfies the condition $\epsilon\leqq\psi_n(l)\leqq\pi-\epsilon$, then according to a previous note there is at least one configuration C_n for the wake problem of P_n. The C_n's form a sequence. The author states that this sequence converges uniformly to the configuration C of the contour BC. Therefore the wake problem for BC has at least one solution. *H. S. Tsien* (Pasadena, Calif.).

Gröbner, Wolfgang. Über eine Näherungsmethode für die ebene Potentialströmung einer kompressiblen Flüssigkeit. Luftfahrtforschung **20**, 184–191 (1943). [MF 9765]

The paper is concerned with the two-dimensional irrotational flow of a compressible fluid around a given obstacle. Using polar coordinates r, θ, the author assumes that, at a sufficiently great distance from the obstacle, the velocity components are represented by convergent series of the form

$$u_r=U[\cos\theta+A_1/r+A_2/r^2+\cdots],$$
$$u_\theta=-U[\sin\theta+B_1/r+B_2/r^2+\cdots],$$

where the coefficients A and B are periodic functions of the angle θ with the common period 2π. The conditions of incompressibility and irrotationality lead to differential equations which permit of determining the coefficients A and B by recursion. It is found that not only A_1 must vanish, but also B_1, which means that only flow patterns with vanishing circulation can be represented in this form. In the case of a circular obstacle the first five terms of the series for u_r and u_θ are shown to give a satisfactory approximation for Mach numbers up to 0.4. *W. Prager.*

Vasilesco, Florin. Recherches théoriques sur les écoulements aérodynamiques à trois dimensions. J. Math. Pures Appl. (9) **21**, 155–198 (1942). [MF 9302]

This paper considers the 3-dimensional, steady, irrotational, discontinuous flow of an incompressible fluid in the case where the motion is identically the same in every half-plane zOr through an axis $z'Oz$, where the velocity vector at each point P in space is in the half-plane zOr through P and where the velocity at infinity is constant. The author's object is to determine the fluid surfaces along which moving fluid slips past fluid at rest. Call jet surfaces Λ such of these fluid surfaces which extend to infinity (for example, the surface of the wake behind an obstacle), and call slip surfaces Λ' such of these surfaces which border upon bounded regions of separation. Let (λ) and (λ') denote the meridian sections of Λ and Λ' by an arbitrary half-plane zOr.

The principal results are the following. (1) Every line (λ) is convex toward the current. (2) A necessary and sufficient condition is given that a set of lines (λ) and (λ') correspond to a flow of the above type. (3) For each 3-dimensional flow there exists an artificial 2-dimensional flow such that the lines (λ) and (λ') are the same for the two flows. Thus the 3-dimensional problem is reduced to 2 dimensions, but with the difficulty that in the 2-dimensional flow the velocity along the lines (λ) and (λ') is not constant. (4) Let θ be the inclination of the velocity vector to the z-axis, and let Ψ be an arbitrary harmonic function subject to certain boundary conditions. Green's theorem expressed in terms of θ and Ψ, that is,

$$\int\int_d(\theta\Delta\Psi-\Psi\Delta\theta)dzdr+\int_c(\theta d\Psi/dn-\Psi d\theta/dn)ds=0,$$

is reduced (in the case of a single obstacle intersecting the z-axis) to an integral equation whose solution determines (λ) for a 2-dimensional flow. (5) The result (4) is extended to 3 dimensions for the case where the obstacle is a sphere. *C. C. Torrance* (Cleveland, Ohio).

Vasilesco, Florin. Sur les mouvements avec sillage. C. R. Acad. Sci. Paris **215**, 317–319 (1942). [MF 9504]

This report outlines another method for determining the jet lines (λ) [cf. the paper reviewed above]. *C. C. Torrance* (Cleveland, Ohio).

Godefroy, Marcel. Sur le mouvement des lignes de discontinuité de vitesse dans un liquide. C. R. Acad. Sci. Paris **212**, 1079–1080 (1941). [MF 9640]

Let w denote complex velocity, let the point $\zeta(\sigma)=\xi(\sigma)+i\eta(\sigma)$ trace the path (C) as σ varies from -1 to $+1$ and let $R(\sigma)$ be real over $-1\leqq\sigma\leqq+1$. The object of this paper is to parametrize the relation

$$w(z)=\frac{1}{2\pi i}\int_{-1}^{+1}\frac{R(\sigma)d\sigma}{z-\zeta(\sigma)}$$

so that the pressure is not subject to discontinuity upon crossing (C). The author carries out this parametrization (under certain conditions) in such a way that the preceding relation reduces to an integro-differential equation for ζ. *C. C. Torrance* (Cleveland, Ohio).

Ballabh, Ram. Superposable motions in a heterogeneous incompressible fluid. Proc. Benares Math. Soc. (N.S.) **3**, 1–9 (1941). [MF 9431]

The equations of a viscous, heterogeneous but incompressible fluid are considered from the point of view of the author's earlier definition of "superposability" [same Proc. (N.S.) **2**, 69–79 (1940); these Rev. **3**, 283]. General necessary and sufficient conditions are obtained. The most general self-superposable solution is obtained in which the vortex vector always has the same direction as the velocity and in which the density ρ depends only upon time and a single spatial coordinate z. It turns out that ρ must then be still further specialized. Also the magnitude of the velocity is the same throughout the liquid at any instant, while its direction is the same for all particles lying in any plane parallel to $z=0$. The stream lines are straight lines lying in the planes $z=$ constant. *D. C. Lewis* (New York, N. Y.).

Dietz, D. N. A new method for calculating the conduct of translation waves in prismatic canals. Physica **8**, 177–195 (1941). [MF 9543]

The formula of the wave velocity has been improved by adding two resistance terms:

$$c=\sqrt{(gh)}\left\{1+\frac{3}{2}\eta-\frac{g\sqrt{(gh)}}{2C^2h}\left(\int Adt+B\frac{dt}{d\eta}\right)\right\}.$$

Without the influence of the resistance each positive wave would become steeper at its front and would be flattened at its back. The influence of the resistance appears mainly from the term

$$-\frac{g\sqrt{(gh)}}{2C^2h}B\frac{dt}{d\eta}.$$

This retards the front, the back is accelerated and this the more so the smaller the gradient is of the surface.

Author's summary.

Scherberg, Max G. Regions of infinite acceleration and flow realms in a compressible fluid. J. Aeronaut. Sci. **10**, 223–226, 231 (1943). [MF 8745]

The author, following von Kármán [J. Aeronaut. Sci. **8**, 337–356 (1941); these Rev. **3**, 220], associates regions of infinite acceleration with regions of shock waves in the flow of a compressible fluid. The author develops further conditions for the absence and the presence of regions of infinite acceleration, which were earlier given some treatment by von Kármán [loc. cit.]. As the Kutta theory introduced circulation to explain away infinite acceleration at the trailing edge of a lifting airfoil, shock wave theory was introduced to explain away infinite acceleration in compressible fluids. From the general equation of the motion of a particle of a compressible fluid, a formula for the acceleration is obtained. By suitably choosing a reference system, a simplified form of the general acceleration equation is investigated for the conditions which do not give rise to infinite acceleration. Certain physical interpretations are drawn from these results. Three known simple flows are examined to illustrate the general results. *A. Gelbart.*

Meixner, J. Reversible Bewegungen von Flüssigkeiten und Gasen. Ann. Physik (5) **41**, 409–425 (1942). [MF 9234]

This paper investigates the motions of liquids and gases subject to the reversibility condition $\vartheta=0$, where $\vartheta=\rho dS/dt+\operatorname{div}\vec{S}$, where S is the entropy and ρ the density. The simplest motions with $\vartheta=0$ are such that the fluid acts like a rigid body having a constant acceleration and a constant angular velocity. These are the only motions possible when the volume viscosity is not equal to 0, and in this case the condition $\vartheta=0$ is equivalent to Eckart's criteria for thermodynamic equilibrium [C. Eckart, Phys. Rev. (2) **58**, 267–275 (1940)] except for the conditions $\partial\bar{v}/\partial t=0$ and grad $p=0$. However, if the volume viscosity equals 0, there exist other motions with $\vartheta=0$, such as a radial adiabatic expansion. These motions the author investigates in detail in the case of mixtures of two ideal gases and in the case of arbitrary liquids and gases with one component. These motions represent reversible processes which involve changes in density and temperature, they are executed in finite lengths of time and they do not satisfy Eckart's criteria. *C. C. Torrance* (Cleveland, Ohio).

Millionshtchikov, M. D. On the theory of homogeneous isotropic turbulence. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 615–618 (1941). [MF 9614]

One of the main tasks of any statistical theory of isotropic turbulence is the determination of the correlations between the velocity fluctuations. Due to the nonlinear inertia terms in the Stokes-Navier equations it is, however, impossible to obtain an equation for the correlation of the order n without involving the correlation of the order $n+1$. The author proposes to compute approximate expressions for the third moments (or correlations) along the following line. (i) The second moments are given approximately by a normal distribution. (ii) For the Gaussian distribution the relation between fourth and second moments is known. The third order correlations are zero. (iii) The equation of motion furnishes a relation from which the third moments can be computed from the approximate expression for the fourth moments. The author finds that the asymmetry of the distribution expressed in the magnitude of the third moments relative to the second moments varies inversely with the fourth root of the time. *H. W. Liepmann.*

Millionshtchikov, M. D. On the rôle of third moments in isotropic turbulence. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 619–621 (1941). [MF 9615]

In a previous paper [see the preceding review] the author obtained approximate expressions for the third moments. Using von Kármán and Howarth's equation, he computes the perturbation on the second moments due to the triple correlation. It is found that this correction to the second moments is most pronounced near the origin of the correlation curve, that is, for small distances between the points at which the fluctuations are taken. The relative size of the correction varies inversely with the $\frac{3}{4}$ power of the time. The relative correction is found to increase if the initial turbulence level is increased and decreases with increasing viscosity. This fact is, of course, due to the corresponding change in relative magnitude of the inertia and viscous terms in the equations of motion.

H. W. Liepmann (Pasadena, Calif.).

Rocard, Yves et Véron, Marcel. Sur la convection vive d'un fluide s'écoulant en régime turbulent permanent le long d'une plaque. C. R. Acad. Sci. Paris **215**, 402–404 (1942). [MF 9508]

As in a previous note [C. R. Acad. Sci. Paris **214**, 301–304 (1942); these Rev. **4**, 176], the authors are concerned with the effect of chemical reaction on heat transfer in a boundary layer. They call this effect the "forced" or "active convection" in differentiation from the ordinary convection which they call the "stagnant convection." In this note they consider a turbulent boundary layer along a flat plate. If y is the distance perpendicular to the plate, l the mixture length, ρ the density, $v'=y-$component of the fluctuating velocity, c the specific heat, T the temperature, $\pm Q_f$ the heat generated by complete reaction of molecules originally contained in a unit volume, β the fraction of original molecules transformed by reaction, u the mean velocity along the plate, then the heat transfer by "stagnant convection" is

$$Q_1=(\rho v'l)c\partial T/\partial y.$$

The heat transfer by "active convection" is

$$Q_2=\pm(\rho v'l)(Q_f/\rho)\partial\beta/\partial y.$$

The shear is $\tau=(\rho v'l)\partial u/\partial y$. If $Q=Q_1+Q_2$, then

$$\frac{Q}{c\tau}=\left(\frac{\partial T}{\partial y}\pm\frac{Q_f}{c\rho}\frac{\partial\beta}{\partial y}\right)\Big/\frac{\partial u}{\partial y}.$$

This is then an extension of Reynolds' analogy to include "active convection." By using various approximations, different formulae connecting skin friction coefficient and heat transfer coefficients are obtained. The results can always be separated into an active convection part and a stagnant convection part. The former may be the predominating part for a gas flame. *H. S. Tsien* (Pasadena, Calif.).

Roy, S. K. Motion of a local vortex round a disturbed corner. Proc. Benares Math. Soc. (N.S.) **3**, 71–93 (1941). [MF 9438]

This paper is a sequel to a series of investigations made by Miyadzu [Philos. Mag. (7) **16**, 553 (1933); (7) **17**, 1011 (1934); (7) **19**, 652 (1935); (7) **25**, 567 (1938)] on the two

dimensional motion of a vortex around a corner with the angle π/n formed by two straight planes. The fluid is assumed to be incompressible and nonviscous. The author considers the effect of putting a small circular hump with center at the vortex of the corner. The flow region is first transformed into the upper half of the t-plane. Besides the vortex, there is a flow given by the complex potential $w=-Ut^m$. The calculations are made with the aid of the "Kirchhoff-Roth path function." If a is the radius of the circular hump, ρ the distance of the vortex from the vortex, $\beta=(a/\rho)^n$, then the author first shows that the first order effect of the small circular hump on the vortex motion is proportional to β^2. The vortex tends to move away from the corner as expected. The equilibrium positions of the vortex are given by the points where the path function has a stationary value. For the case $m=1$, the presence of the small circular hump introduces a new set of equilibrium positions besides the usual ones on the bisecting line of the corner. If $n>1$, the locus of the new equilibrium positions is a closed curve, while, if $n<1$, the locus is divided into two branches lying on the two sides of the circular hump. For $m=2$, no equilibrium position exists. For $m=3$, there are equilibrium positions off the bisecting line of the corner only if $n>3$. With increase in β, the lowest value of n for this type of equilibrium position also increases. The question of stability of the equilibrium positions is investigated by the author only for those on the bisecting line of the corner. The presence of the circular hump tends to make them less stable. The paper concludes with the demonstration that other types of protuberance have the same effect as the circular hump for distances not too small compared with the size of the protuberance. *H. S. Tsien.*

Stepanyantz, L. G. The calculation of laminar boundary layers around bodies of revolution. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 317–326 (1942). (Russian. English summary) [MF 7584]

Loitsyansky [J. Appl. Math. Mech. **5**, 453–470 (1941); these Rev. **4**, 120] replaces the von Kármán momentum equation in the boundary layer theory by Jourdain's variational principle, by which

$$\int_\tau (\rho\mathbf{w}-\mathbf{F}+\mathrm{grad}\ p+\mu\nabla^2\mathbf{v})\cdot\delta\mathbf{v}\,d\tau=0.$$

Here $\mathbf{v}$ is the velocity vector, $\mathbf{w}$ the acceleration vector, $\mathbf{F}$ the vector of the volume forces and $d\tau$ the volume element; ρ, p and μ are the density, the pressure and the viscosity coefficient, respectively. Neglecting $r^{-1}(\partial r/\partial y)(\partial u/\partial y)$, the author obtains, in the case of an axial symmetric flow,

$$\int_0^h r[u(\partial u/\partial x)+v(\partial u/\partial y)-UU'-\rho^{-1}\mu(\partial^2 u/\partial y^2)]\delta u\,dy=0.$$

Here $r=r(y)$ gives the boundary curve of the domain around which the motion takes place. Introducing two different assumptions $(u/U)=\sum_{k=0}^n a_k(y/h)^k$, $n=2$ and 4, the author obtains two formulae for the thickness of the boundary layer h. (One is less, the other more, exact.) The results are compared with the experimental data. *S. Bergman.*

Sédille, Marcel. Sur l'influence de l'allongement dans les écoulements plans limités par deux plans parallèles, et sur la constitution des couches limites de ces plans. C. R. Acad. Sci. Paris **213**, 641–643 (1941). [MF 9647]

The author discusses the influence of the aspect ratio of cylindrical bodies in fluid flow. For the case of flow through a lattice of airfoils of finite aspect ratio the loss ζ is found approximately to follow the law:

$$\zeta=\zeta_\infty+\mathrm{const.}/x,$$

where x is the aspect ratio and ζ_∞ the loss for two dimensional flow, that is, for $x\to\infty$. *H. W. Liepmann.*

Lehner, J. and Mark, C. An application of the method of the acceleration potential. Quart. Appl. Math. **1**, 250–261 (1943). [MF 9365]

The authors determine the results of linearized thin-airfoil theory for a wing-aileron combination separated by a finite gap. Stationary flow and equal chord for wing and aileron are assumed. For stationary problems the acceleration potential method amounts to formulating the problem in terms of the derivative in main flow direction of the velocity potential. Taken over from the velocity potential procedure are the results pertaining to the nature of the singularity at the leading edge of the profile. The solution of the present problem involves elliptic functions. Reference should be made to the solution of this and more general problems by integral equation methods by Söhngen [Math. Z. **45**, 245–264 (1939)]. *E. Reissner* (Cambridge, Mass.).

Schmitt, Pierre. Contribution à l'étude de l'écoulement autour de deux profils d'aile. C. R. Acad. Sci. Paris **215**, 400–401 (1942). [MF 9507]

The problem of two-dimensional flow around two airfoils can be divided into two parts: (1) the determination of the flow around two circles wih proper circulations; (2) the conformal transformation of the circles into airfoils. The first part of the problem was solved by M. Lagally [Z. Angew. Math. Mech. **9**, 299–305 (1929)]. The author solves the second part of the problem by applying an ordinary airfoil transformation first to one circle. Then the first circle is transformed into an airfoil but the second circle is only slightly deformed. Then the whole figure is again transformed by using an ordinary transformation to deform the second "circle" into an airfoil. During the second transformation the first airfoil is only slightly disturbed for usual proportions of the airfoils. It should be noted, however, that this same procedure was investigated in detail by M. P. Dupont quite some time ago [Proc. 3rd Internat. Congr. App. Mech., vol. 1, Stockholm, 1931, pp. 372–386]. *H. S. Tsien* (Pasadena, Calif.).

Kotchin, N. E. Contribution to the theory of a finite span wing circular in plane. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 611–614 (1941). [MF 9613]

This note is the summary of the author's complete paper [Appl. Math. Mech. [Akad. Nauk. SSSR. Prikl. Mat. Mech.] **6**, 287–316 (1942); these Rev. **4**, 228]. The statement of the problem and the method of solution have already been given in the review of that paper. The same problem has also been solved by Th. Schade and K. Krienes [Luftfahrtforschung **17**, 387–400 (1940); **19**, 282–291 (1942); these Rev. **3**, 285; **4**, 177], using the method of acceleration potential. [By comparing Kotchin's approximate numerical results with those of Krienes, it is seen that Kotchin's calculation applies only to the cases of very low frequencies of oscillation and is, in fact, quite close to the values given by C. Possio [Aerotecnica **18**, 1323–1351 (1938)] who assumes both low frequency and large aspect ratio. For applications to the flutter of airplane wings, the main interest in an oscillating circular wing is to determine the accuracy of Possio's theory for this critical case of very

low aspect ratio. From this point of view, Schade and Krienes' investigation, which gives the exact numerical values, is more satisfactory. However, Kotchin also calculates the induced drag of the oscillating circular wing and obtains the very interesting result that the average induce drag for angular oscillation is larger, while the average induced drag for vertical oscillation is smaller than that of a steady wing.] *H. S. Tsien* (Pasadena, Calif.).

v. Borbély, S. Über die Luftkräfte, die auf einen harmonisch schwingenden zwei-dimensionalen Flügel bei Überschallgeschwindigkeit wirken. Z. Angew. Math. Mech. **22**, 190–205 (1942). [MF 8985]

This paper is concerned with the two-dimensional linearized theory for an airfoil oscillating in a fluid which moves at supersonic speed. The differential equation of the problem is linear and hyperbolic. The author obtains the solution which satisfies all the conditions of the problem by the use of indented contour integrals. With the help of the solution formulas are derived for aerodynamic force and moment on the airfoil. These formulas contain elementary functions and the function

$$g(w,\lambda)=\int_0^\infty e^{-iu}J_0(u\lambda)du,$$

for which an infinite series development is given. No numerical results are included. *E. Reissner.*

Helmbold, H. B. und Keune, F. Beiträge zur Profilforschung. V. Theorie des Singularitätenverfahrens. Luftfahrtforschung **20**, 192–206 (1943). [MF 9764]

Helmbold, H. B. und Keune, F. Beiträge zur Profilforschung. VI. Zweite Näherung zur Berechnung der Geschwindigkeitsverteilung nach dem Singularitätenverfahren. Luftfahrtforschung **20**, 196–206 (1943). [MF 9884]

Parts V and VI of an extensive paper concerned with the development of a family of airfoil sections and the computation of the velocity distribution along the profiles [Luftfahrtforschung **20**, 77–80, 81–96, 152–170 (1943); these Rev. **5**, 22; IV not yet published]. A first approximation to this velocity distribution has been given in part III. The present installments deal with the possibilities of improving on these results by a method of successive approximations. Particular attention is given to the study of the velocity distribution at the leading end, where the first approximation leads to infinite velocities. *W. Prager.*

Price, H. L. The lateral stability of aeroplanes. IV. A new geometrical system of analysis. Aircraft Engrg. **15**, 281–287 (1943). [MF 9596]

[The first three installments appeared in the same volume, pp. 193–198, 228–233, 265–267; cf. these Rev. **5**, 81.] The author presents stability diagrams of the type described in part II, covering ranges of angle of attack, aspect ratio and moment of inertia. A comparison is made between previous stability representations and the one proposed by the author. The effects of various parameters are summarized and the contributions of power and flaps are discussed. *W. R. Sears* (Inglewood, Calif.).

Price, H. L. The lateral stability of aeroplanes. V. Disturbed and controlled motions. Aircraft Engrg. **15**, 325–329 (1943). [MF 9597]

[Cf. the preceding review.] The author describes the nature of the motions associated with the various modes. Attention is turned toward the amplitudes of motion as well as the damping as pertinent criteria for the description of airplane stability characteristics. He illustrates the utility of the Heaviside operational methods in the treatment of control problems for stable and unstable airplanes. *W. R. Sears* (Inglewood, Calif.).

Schubert, R. Zur Berechnung der statischen Längsstabilität im Motorflug. Luftfahrtforschung **19**, 271–281 (1942). [MF 7901]

The author investigates the practically important problem of how to predict the static longitudinal stability at power-on condition from the more easily measured longitudinal stability at power-off condition. The effect of engine-propeller combination on stability is separated into the following items: (1) the moment due to the thrust and side force of the propeller, (2) the increase of dynamic pressure and the rotation of the slip-stream at the wind and the tail, (3) the change in down-wash angle due to the velocity of the slipstream, (4) the change in down-wash angle due to the side force of the propeller and (5) the change in down-wash angle due to the change in lift of the wing by the action of the propeller. Each of these items is considered separately and the effects added to give the desired result. The side force of the propeller is calculated by a method due to Harris and Glauert [Durand-Glauert, Aerodynamic Theory, vol. 4, Springer, Berlin, 1935, p. 351]. However, it should be noted that the method is now improved by Rumph, White and Grummann [J. Aeronaut. Sci. **9**, 465–470 (1942)]. Furthermore, if the tail surfaces are situated at the boundary of the slip-stream where a large velocity gradient exists, the section characteristics of those surfaces may be quite different from that measured in a uniform flow [cf. Tsien, Quart. Appl. Math. **1**, 130–148 (1943); these Rev. **5**, 21]. This effect is not considered by the author. *H. S. Tsien.*

Sédille, Marcel. Sur la similitude des turbomachines à fluides compressibles. C. R. Acad. Sci. Paris **213**, 682–685 (1941). [MF 9652]

The author presents a discussion of the dimensionless variables by means of which a compressor can be characterized and classified. The necessity of introducing new parameters such as the Mach number, the ratio of the specific heats and the Prandtl number for the case of large compression ratios is pointed out. *H. W. Liepmann.*

Stewart, H. J. Periodic properties of the semi-permanent atmospheric pressure systems. Quart. Appl. Math. **1**, 262–267 (1943). [MF 9366]

An atmosphere of constant density is considered. Vertical velocities are neglected and it is assumed that the apparent acceleration is negligible compared with the Coriolis acceleration. The effects of friction and of the variation of the Coriolis parameter with latitude are also neglected, so that the fluid motion is that on a rotating disc and not on a rotating sphere. Under these conditions a system of three vortices (anticyclones), symmetrically placed around the center of rotation, is shown to have small "westward" drift in the "northern hemisphere." The small oscillations of the system about its position of equilibrium are worked out for the cases corresponding to anticyclones on a rotating globe at 37° and 29° of latitude, respectively. The normal modes show a short period (2000

to 5000 days) and a long period (70,000 to 250,000 days) and the path of a vortex of the system is always an ellipse.
G. C. McVittie (London).

Lyra, G. Theorie der stationären Leewellenströmung in freier Atmosphäre. Z. Angew. Math. Mech. **23**, ~~1–55~~ 1–28 (1943). [MF 9782]

The steady flow of air over an orographic obstacle is treated as a two-dimensional problem in a vertical plane. The effects of the earth's rotation, of friction, turbulence, radiation and moisture are neglected, but the atmosphere is considered as a compressible fluid with stable stratification. It is further assumed that the perturbations of the flow caused by the topography are so small that the atmospheric perturbation equations can be applied. For the vertical component of motion a differential equation of elliptic type is obtained. If the atmospheric stratification is not isothermal, the coefficients of the equation will depend on the altitude, but for practical purposes they may be treated as constants. The problem is then to find a solution of the wave equation with given values for $z=0$ and vanishing at infinity. The general solution of this boundary value problem is obtained by three different methods. Finally, the upward currents and streamlines over certain types of profiles (infinite and rectangular plateaus and small obstacles) are considered. At lower levels an upward current is found in front of the obstacle (upslope current). Behind the current the disturbances repeat themselves. The magnitude of the disturbances decreases downwind, more rapidly at lower than at higher levels. There is also a certain phase difference between the perturbations at different altitudes. *B. Haurwitz* (Cambridge, Mass.).

Rothstein, W. Strömungen über Bodenerhebungen auf der rotierenden Erde. Z. Angew. Math. Mech. **23**, 72–80 (1943). [MF 9864]

The author investigates the flow of air around mountains of finite lateral extent. The effect of the earth's rotation is taken into account but the atmosphere is assumed incompressible and the influence of friction is neglected. The problem requires the solution of an integral equation which is obtained in the form of a Neumann series. As examples stream lines around mountains of circular and elliptical symmetry are given. These show the deflection of the stream lines to the left (from the direction of flow) when crossing the mountain. *B. Haurwitz* (Cambridge, Mass.).

Zeldovich, J. and Leipunskiy, O. On the propagation of shock waves in the water. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **13**, 183–184 (1943). (Russian) [MF 9733]

Obukhov, A. M. On propagation of sound waves in eddying flow. C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 46–48 (1943). [MF 9718]

The equations of acoustics are derived from the equations of hydrodynamics for the case of a small disturbance superimposed on a quasi-stationary incompressible (but not necessarily irrotational) main flow. The equations are linearized by neglecting second-order quantities, and the disturbance velocity is sought in terms of a "quasi-potential" function φ, defined by

$$\partial \mathbf{v}'/\partial t = \text{grad}\,(\partial\varphi/\partial t) + [\text{grad}\,\varphi,\ \Omega_0],$$

where the total velocity is $\mathbf{v}=\mathbf{v}_0+\mathbf{v}'$, $\mathbf{v}'\ll\mathbf{v}_0$, and $\Omega_0=\text{rot}\,\mathbf{v}_0$. The author points out that in the most interesting practical cases, such as propagation of sound in a turbulent atmosphere, the main flow is eddying rather than potential.
W. R. Sears (Inglewood, Calif.).

Stenzel, H. Über die Berechnung des Schallfeldes unmittelbar vor einer kreisförmigen Kolbenmembran. Ann. Physik (5) **41**, 245–260 (1942). [MF 9232]

A circular diaphragm of radius a, set in a wall, vibrates harmonically. The problem is to find the sound pressure at the diaphragm. The writer starts from the known solution

$$p_1 - ip_2 = (2\pi/\lambda)a\int_0^\infty J_0(\lambda\rho)J_1(\lambda a)d\lambda/\mu, \quad \mu=((2\pi/\lambda)^2-\lambda^2)^{\frac{1}{2}},$$

where ρ is the radial coordinate in the diaphragm. The integrands are expanded in series in a straightforward manner, with repeated appeals to Watson's Bessel's functions. The resulting series representations are shown to be suitable for practical computation and some tables of numerical results are included. *D. G. Bourgin.*

Theory of Elasticity

Chatsirevitch, I. H. Application of Weyl's method to the solution of the plane problem of the theory of elasticity. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 197–202 (1942). (Russian. English summary) [MF 7749]

The paper presents a new solution of the plane boundary problem of the theory of elasticity corresponding to a given distribution of external forces. The corresponding three dimensional problem has been reduced by H. Weyl [Rend. Circ. Mat. Palermo **39**, 1–50 (1915)] to an integral equation based on the solution given by Boussinesq for a half plane. The integral equation was solved by Weyl independently of the existence or nonexistence of solutions of the corresponding transposed homogeneous equation. For the two dimensional case, the author shows that the corresponding homogeneous equation does not possess a nontrivial solution. Using Muschelišvili's notations, the problem is reduced to the determination of two analytic functions $\varphi(z)$ and $\psi(z)$ in a simply connected domain S satisfying on the boundary the condition

$$(*)\qquad \varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=i\int_0^s (X+iY)ds$$

with a given $X+iY$. (The author considers throughout the paper multiply connected regions, in which case all formulae are more complicated.) Putting

$$\varphi(z)=(-1/2\pi i)\int\omega(t)\log(t-z)ds$$

and

$$\psi(z)=(-1/2\pi i)\left\{\int\overline{\omega(t)}\log(t-z)ds-\int \bar{t}\omega(t)ds/(t-z)\right\},$$

an integral equation is obtained for $\omega(t)$ by differentiation of (*) with respect to the arc s. It is shown that the resulting integral equation possesses a unique solution.
A. Weinstein (Toronto, Ont.).

Zamyatina, V. N. Two-dimensional problem of the theory of elasticity for circular rings. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 53–74 (1942). (Russian. English summary) [MF 7735]

The first part of the paper presents a solution of the first fundamental two-dimensional problem for a concentric cir-

cular ring, the method being based on a well-known formula of H. Villat [Rend. Circ. Mat. Palermo **33**, 134–174 (1912)]. Previous solutions have been given by Timpe [Z. Math. Phys. **52**, 348 (1905)] and by Muskhelišvili [textbook, 1933]. In the second part the author proves the following theorem. Let $z_1=(\alpha z+\beta)/(\gamma z+\delta)$, $z_1=x_1+iy_1$, $z=x+iy$, and let $w_1(x_1, y_1)$ be a biharmonic function. Then $w=w_1|\gamma z+\delta|^2$ is a biharmonic function of the variables x,y. This theorem is applied to the solution of the same problem as above for domains which can be represented conformally on a concentric ring by means of a fractional linear function, namely, an eccentric ring and a plane with two holes.

A. Weinstein (Toronto, Ont.).

Weber, Constantin. Halbraum mit Halbkugelbelastung. Z. Angew. Math. Mech. **22**, 318–321 (1942). [MF 9119]

Usunoff [Z. Angew. Math. Mech. **22**, 262–269 (1942); these Rev. **5**, 26] has shown that the direct determination of certain stress components in a semi-infinite elastic solid $x<0$ acted upon by a hemispherical distribution of stress normal to the plane $x=0$ depends upon the evaluation of a multiple integral. The evaluation of this integral led Usunoff to elliptic integrals. By use of a method previously developed [Z. Angew. Math. Mech. **20**, 117–118 (1940); these Rev. **2**, 31], the author shows that the stresses can be expressed in terms of elementary functions. *N. Coburn.*

Galin, L. A. Mixed problems of the theory of elasticity involving frictional forces in relation to half-planes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 91–95 (1943). [MF 9721]

This paper treats the elastic half plane with several rigid bodies (stamps) pressed against the boundary so as to produce both normal and tangential forces at the contact. The following three cases are considered: (a) stamps at rest pressed against an isotropic half plane, (b) stamps moving uniformly along the boundary of an isotropic half plane, (c) stamps at rest pressed against a half plane with arbitrary anisotropy. All three cases are reduced to the problem of determining a function $W(z)=U+iV$ analytic in the upper half plane and satisfying the condition $a(x)U+b(x)V=f(x)$ on the real axis, $a(x)$, $b(x)$ and $f(x)$ being prescribed functions; $W(z)$ is determined. In case (a) the reduction is carried out by the method of N. I. Mushelišvili [C. R. (Doklady) Acad. Sci. URSS (N.S.) **3**, 51–54 (1935)] and in case (c) by the method of S. G. Lekhnitsky [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **1**, 282–289 (1933)]. *G. E. Hay* (Ann Arbor, Mich.).

Charrueau, André. Sur les équilibres limites des milieux continus. C. R. Acad. Sci. Paris **213**, 820–822 (1941). [MF 9658]

The author considers a plane plastic material yielding under the Coulomb condition

$$(1)\qquad \sigma_1-\sigma_2=2\{f(2\sigma)\}^{\frac{1}{2}},$$

where σ_1, σ_2 are the principal stresses, σ is the mean normal stress and $f(2\sigma)$ is a differentiable function of σ. Mandel [same C. R. **206**, 317–318 (1937)] has shown that the characteristics of the system of equations consisting of the equilibrium relations (containing body forces) and equation (1) are the slip lines. By use of these slip lines as parameter lines, the equilibrium relations are given in two forms: (1) in terms of x, σ and θ, the angle between the x-axis and the direction of the larger principal stress, as dependent variables; (2) in terms of x, σ_x, σ_{xy}, as dependent variables, where σ_x is the stress component parallel to the x-axis and σ_{xy} is the shear stress. No details of the computation are given. Further, by expressing equation (1) in terms of the stresses σ_x, σ_y, σ_{xy}, introducing an Airy stress function and differentiating equation (1) partially with respect to y, the author shows that equation (1) may be written as a linear second order partial differential equation in four independent variables. *N. Coburn* (Austin, Tex.).

Gogoladze, V. G. On Rayleigh waves on the confines of two solid elastic media. C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 15–17 (1941). [MF 9619]

Shevtchenko, K. N. Oscillation of a plate in its own plane. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 41–52 (1942). (Russian. English summary) [MF 7734]

The author uses direct methods of the calculus of variations in order to establish the existence of the fundamental mode of vibration for a free plate oscillating in its own plane. It is required to find the minimum of the integral

$$(8\mu)^{-1}\int\int(1-2\lambda)(\operatorname{div}\mathbf{u})^2+(u_x-v_y)^2+4(u_y+v_x)^2dxdy$$

with the subsidiary conditions $\int\int \mathbf{u}^2dxdy=1$, $\int\int \mathbf{u}dxdy=0$, $\int\int(u_y-v_x)dxdy=0$. The author shows that a subsequence of a minimizing sequence [not the sequence itself, as incorrectly asserted] converges towards a solution of the corresponding differential problem and satisfies the natural boundary conditions. In the proofs the well-known formulae of Betti, Green's stress tensor and Somigliana's displacement tensor are employed. *A. Weinstein.*

Lourye, A. I. On the theory of thick plates. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 151–168 (1942). (Russian. English summary) [MF 7746]

Let

$$\nu\frac{\partial\theta}{\partial x}+\left(\frac{\partial^2}{\partial z^2}+\Delta\right)u=0,$$

etc., be the equations for the displacements u, v, w of a plate bounded by $z=\pm h$. The author integrates these equations with respect to z, considering for a moment the symbols $\partial/\partial x$, $\partial/\partial y$ and $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$ as "constants." Using the boundary conditions, the author reduces the problem to the integration of an equation of infinite order $F(\Delta)=q$, where F is an entire function and q denotes the lateral load. A detailed discussion of the solutions of this equation is given and the results are applied to the bending of a thick round plate. *A. Weinstein* (Toronto, Ont.).

Federhofer, Karl. Berechnung der Auslenkung und Spannungen beim Kippen des geschlossenen Kreisringes. Z. Angew. Math. Mech. **23**, 35–47 (1943). [MF 9783]

Previous investigations have determined the critical uniform radial compressive stress for the lateral buckling of a closed circular ring under certain assumptions as to the direction of the lines of action of the compressive forces after buckling. The present paper is devoted to determining the magnitudes of the lateral deflection and the stresses after lateral buckling has occurred. This is done on the basis of each of two assumptions as to the directions of the lines of action of the compressive forces after buckling has occurred.

H. W. March (Madison, Wis.).

Wegner, U. Neues Verfahren zur Berechnung der Spannungen in Scheiben. Z. Verein. Deutsch. Ingenieure **87**, 443–444 (1943). [MF 9216]

A method similar to the Ritz method is described for the determination of stresses in a plate under loads in the plane of the plate. The Airy stress function is obtained from the potential function which minimizes a certain integral. This integral is the strain energy of the analogous plate under lateral loading. The integrand contains the potential function and another function which must be chosen beforehand to satisfy the boundary conditions. The stress concentration due to a semicircular notch in a semi-infinite plate under tension is given as an example. *P. W. Ketchum.*

Kufareff, P. Berechnung der Spannungen im anisotropen Keil. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 534–535 (1941). [MF 9608]

This paper deals with plane stress in an anisotropic infinite wedge loaded on the faces. The method employed is that of S. G. Lekhnitsky [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **1**, 282–289 (1933)], in which the problem is reduced to the determination of two functions of a complex variable, analytic within the wedge and satisfying certain boundary conditions. *G. E. Hay.*

Kufareff, P. und Sweklo, W. Bestimmung der Spannungen im anisotropen Streif. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 609–610 (1941). [MF 9612]

This paper deals with plane stress in an anisotropic infinite strip loaded on the edges. The procedure follows that in a previous paper by one of the authors [cf. the preceding review]. *G. E. Hay* (Ann Arbor, Mich.).

Föppl, Ludwig. Dünnwandige Hohlzylinder gleicher Festigkeit gegen Innen- und Aussendruck. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1942**, 71–80 (1942). [MF 9804]

The paper is concerned with the strength of certain twin containers. The cross section of these thin-walled cylindrical containers is formed by two circular arcs (of central angles greater than 180°) which lie on opposite sides of a common chord. The outer walls follow these circular arcs; a web following the chord separates the container into two cells. It is shown that, for a given factor of safety, such twin containers with equal pressures in the two cells can be designed so as to absorb not more material than a single cylindrical container of circular cross section which possesses the same volume and is subject to the same interior pressure. *W. Prager* (Providence, R. I.).

Galerkin, B. G. Stability of a cylindrical shell. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 49–56 (1943). (Russian. English summary) [MF 9725]

The paper is concerned with the problem of stability of cylindrical shells subjected to a distribution of loads normal to the surface of the shell. In an earlier paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **4**, 270–275 (1934)] the author has shown that the determination of critical loads in a cylindrical shell, with circular cross-section, can be reduced to the solution of a single partial differential equation of the eighth order for the displacement function. The author specializes his equation and investigates critical loads for the following problems: (a) stability of a ring subjected to a uniform pressure; (b) stability of a circular cylindrical shell whose lateral surface is free of loads and which is compressed by a uniform load distributed over the ends; (c) Euler's critical load problem; (d) stability of a cylindrical shell, simply supported at the ends, and compressed by a uniform load distributed over the lateral surface. *I. S. Sokolnikoff* (Madison, Wis.).

Sokolovsky, W. W. Equations of momentless shells. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 57–64 (1943). (Russian. English summary) [MF 9726]

The paper is devoted to a study of the equilibrium equations of momentless shells, and includes a discussion of the following special cases: (a) shells whose middle surfaces are surfaces of revolution (the known solution obtained by H. Reissner for a spherical shell appears as a special case); (b) shells whose middle surfaces are central quadrics; (c) shells in the form of an elliptical paraboloid; (d) conical and cylindrical shells. *I. S. Sokolnikoff* (Madison, Wis.).

Novojilov, V. V. On an error in a hypothesis of the theory of shells. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 160–164 (1943). [MF 8682]

The author discusses the approximations customarily made in the theory of bending and stretching of thin elastic shells. He offers arguments tending to show that when the hypothesis is made that the normal to the undeformed middle surface is deformed into the normal to the deformed middle surface nothing is gained by refining the stress resultant strain relations which are generally used in shell analysis. *E. Reissner* (Cambridge, Mass.).

Novojilov, V. V. On the solution of thin shell theory problems in stresses and moments. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 294–297 (1943). [MF 9579]

Some consequences are discussed of a form of the equations of the theory of thin elastic shells which has been given by W. Sokolovsky [same C. R. (N.S.) **16**, 19–24 (1937)]. *E. Reissner* (Cambridge, Mass.).

Gorbunov, B. N. The calculation of a space frame with thin-walled bars. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 65–70 (1943). (Russian. English summary) [MF 9727]

The author introduces the concepts of a unitary angular displacement of a joint and of a regular deformation of a bar and suggests a method of calculation by using these concepts. *I. Opatowski* (Chicago, Ill.).

Craemer, H. Der Momentenausgleich plastischer Balkentragwerke und die Verträglichkeit der Formänderungen. Ing.-Arch. **13**, 285–292 (1943). [MF 9072]

The author considers a material which follows Hooke's law in the elastic domain and the Tresca condition in the plastic domain. In particular, the author poses the following question: does the total moment expression for the elastic and plastic regions of a beam correspond with the assumed geometric properties of the deformation along the beam? In order to discuss this question, the moment expression for an I beam in torsion is developed. It is assumed that (1) the upper portion of the top flange is plastic; (2) the lower portion of this flange is elastic; (3) the lower flange behaves in the same manner; (4) the flanges are connected by a thin rigid rod. By introducing two parameters which vary with the length of the flange and the depth of the plastic layer in the flange, respectively, the author finds the

moment expression for all beams which vary in type from the rectangular bar to the "ideal" I beam, and whose state varies from the purely elastic to the completely plastic. Graphical comparisons are made. Further, the moment equation for an I beam whose flanges and connecting rod are completely plastic is derived. [Equations 5 and 6 have been omitted from the paper.]

In the second section the author applies his moment expression to a built-in horizontal beam on which two concentrated forces of magnitude P act at points one-third and two-thirds of the distance from either wall. It is shown that the moment expression does not correspond with the geometric properties of the deformation. Examining the work of Prager [Bauing **14**, 65–67 (1933)], Hohenemser [Ing.-Arch. **2**, 472–482 (1931)] and Fritsche [Z. Angew. Math. Mech. **11**, 176–191 (1931)] in connection with similar problems, it is shown that contradictions exist between the moment expression and the deformation. In order to resolve this contradiction, the author assumes that each force of support is not concentrated but is distributed over a small length of the built-in bar. By studying an example, the author shows that this assumption leads to conclusions for which the moment expression and the deformation are compatible. *N. Coburn* (Austin, Tex.).

Cox, H. Roxbee. A two-dimensional approach to three-dimensional framework problems. J. London Math. Soc. **18**, 20–23 (1943). [MF 9206]

The author points out that the analysis of the completeness or redundancy of a three-dimensional framework is often simplified by the use of the conception of the "boundary polygon." In its simplest form, this is defined as a closed circuit, formed by bars of the framework, passing only once through each joint. If such a boundary polygon consisting exclusively of bars of the given framework does not exist, suitable "shadow trios" of bars may be added so that the thus enlarged framework possesses a boundary polygon. A two-dimensional representation of the framework can then be obtained by deforming the boundary polygon into a plane convex polygon, the remaining bars forming a bracing of this polygon. The following theorem is found useful in obtaining simply-stiff frameworks. A two-dimensional arrangement of bars representing a simply-stiff space framework can be formed by coalescing two equal polygons of boundary bars in each of which there is an arrangement of bracing bars which, with its polygon, forms a simply-stiff plane framework, provided that these two frameworks have neither a bracing bar nor, when imagined free to collapse in three dimensions, a degree of freedom in common. *W. Prager* (Providence, R. I.).

Gaskell, R. E. On moment balancing in structural dynamics. Quart. Appl. Math **1**, 237–249 (1943). [MF 9364]

This paper extends to certain dynamical problems the method of moment balancing introduced by Hardy Cross [Trans. Amer. Soc. Civil Engrs. **96**, 1 (1932)] and further developed by others for the analysis of statical problems of continuous beams and frameworks. In this extension to dynamical problems the forced vibrations of such systems are considered. The possibility of the corresponding extension of the method of balancing changes of angle [Grintner, L. E., Trans. Amer. Soc. Civil Engrs. **102**, 1020 (1937)] is pointed out and illustrated by examples. The process of moment balancing is shown to converge if the frequency of the forced vibration is less than the smallest natural frequency of the structure. The process of balancing of changes of angle is not always convergent under the same condition. *H. W. March* (Madison, Wis.).

Pipes, Louis A. Applications of the operational calculus to the theory of structures. J. Appl. Phys. **14**, 486–495 (1943).

An elementary exposition of the use of the Laplace transform for the solution of certain two point boundary value problems of elasticity. The deflections of elastic cords and beams, both with and without elastic foundations, and with various types of lateral loading (distributed loads, concentrated loads, couples, etc.), are worked out in detail by this method. The use of the Laplace transform in such cases reduces, but does not entirely eliminate, the number of arbitrary constants to be evaluated. The paper furnishes a clear and rapid introduction to applications of the Laplace transform which could be used in beginning courses in the formal solution of differential equations. An appendix contains a small table of transforms.

P. W. Ketchum (Urbana, Ill.).

Leibenson, L. S. Application of variation equation of elasticity to the solution of a geomechanic problem of earth deformation and vibration. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1942**, 195–204 (1942). (Russian. English summary) [MF 8085]

[The correct translation of the title reads: An application of a variational equation in the theory of elasticity to the solution of the problem in geomechanics on the deformation and on the vibration of the earth.] The author investigates an important problem of geomechanics, namely, the problem of deformation of the earth sphere by forces which possess a harmonic potential. The sphere is supposed to consist of an incompressible fluid subject to the law of gravitation. The author assumes that the initial stress distribution is static. His method is based on the calculus of variations. He obtains a simple estimate of the influence of the variation of density and of the variation of elasticity and applies the method to the question of the frequency of free spherical vibrations of the earth. An estimate of the influence of the gravitation on the length and period of vibration is given.

S. Bergman (Providence, R. I.).

Richter, C. F. Mathematical questions in seismology. Bull. Amer. Math. Soc. **49**, 477–493 (1943). [MF 8764]

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